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Contents

Modal logic with the difference modality of topological T_0-spaces

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Abstract

The aim of this paper is to study the topological modal logic of T_0 spaces, with the difference modality. We consider propositional modal logic with two modal operators $\Box$ and $[\neq]$. Operator $\Box$ is interpreted as an interior operator and $[\neq]$ corresponds to the inequality relation. We introduce logic $S4DT_0$ and show that $S4DT_0$ is the logic of all T_0 spaces and has the finite model property and decidable.

Keywords: Kripke sematics, finite model property, completeness, topological semantics

1 Introduction

In this paper, apart from $\Box$ we also deal with the difference modality (or modality of inequality) $[\neq]$, interpreted as true everywhere except here. The expressive power of this language in topological spaces has been studied by Gabelaia in [8], where the author presented axiom that defines T_0 spaces. For T_n, where $n \geq 1$ corresponding logics were known (cf. [3], [8]). We proved the completeness of $S4DT_0$ with respect to topological T_0-spaces and showed that the logic has finite model property.

2 Preliminary

Formulas are constructed in a standard way from a countable set of propositional variables PROP, logical connectives $\bot$ (false) , $\to$ and one-place modalities $\Box$ and $[\neq]$. $(\vee, \wedge, \neg, \top, \equiv)$ are expressed as usual and also $\Diamond\phi = \neg\Box\neg\phi$, $\langle\neq\rangle\phi = \neg[\neq]\neg\phi$. We denote $[\neq]A \wedge A$ by $[\forall]A$.

The set of all bimodal formulas is called the *bimodal language* and is denoted by $\mathcal{ML}_2$.

A *normal bimodal logic* is a subset of formulas $L \subseteq \mathcal{ML}_2$ such that

1. L contains all the classical tautologies:

2. L contains the modal axioms of normality:

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$$\Box(p \to q) \to (\Box p \to \Box q),$$
$$[\neq](p \to q) \to ([\neq]p \to [\neq]q);$$

3. L is closed with respect to the following inference rules:

$$\frac{\phi\to\psi,\ \phi}{\psi}\ \text{(MP)},$$
$$\frac{\phi}{\Box\phi},\ \frac{\phi}{[\neq]\phi}(\to\Box, \to[\neq]),$$
$$\frac{\phi}{[\psi/p]\phi}\ \text{(Sub)}.$$

Let L be a logic and Γ be a set of formulas. The minimal logic containing $L \cup \Gamma$ is denoted by $L + \Gamma$. We also write $L + \psi$ instead of $L + \{\psi\}$.

In this paper we will use the following axioms:

($T_\Box$) $\Box p \to p$,
($4_\Box$) $\Box p \to \Box\Box p$,
($D_\Box$) $[\forall]p \to \Box p$,
(B_D) $p \to [\neq]\langle\neq\rangle p$,
(4_D) $[\forall]p \to [\neq][\neq]p$,
(AT_0) $(p \wedge [\neq]\neg p \wedge \langle\neq\rangle(q \wedge [\neq]\neg q)) \to (\Box\neg q \vee \langle\neq\rangle(q \wedge \Box\neg p))$.

We define the following logics:

$$S4 = K_2 + T_\Box + 4_\Box$$
$$S4D = S4 + D_\Box + B_D + 4_D$$
$$S4DT_0 = S4D + AT_0$$

A topological model on a topological space ([6]) $\mathbb{X} := (X, \Omega)$ is a pair $(\mathbb{X}, V)$, where $V : PROP \to P(X)$ (the set of all subsets), i.e. a function that assigns to each *propositional variable* p a set $V(p) \subseteq X$ and is called a *valuation*. The truth of a formula ϕ at a point x of a topological model $\mathcal{M} = (\mathbb{X}, V)$ (notation: $\mathcal{M}, x \vDash \phi$) is defined as usual by induction, particularly

$$\mathcal{M}, x \vDash \Box\phi \Leftrightarrow \exists U \in \Omega(x \in U \text{ and } \forall y \in U(\mathcal{M}, y \vDash \phi)),$$
$$\mathcal{M}, x \vDash [\neq]\phi \Leftrightarrow \forall y \neq x(\mathcal{M}, y \vDash \phi).$$

Let $\mathcal{M} = (X, \Omega, V)$ be a topological model and ϕ be a formula. We say that ϕ is true in the model $\mathcal{M}$ (notation: $\mathcal{M} \vDash \phi$), if it is true at all points of the model, i.e.

$$\mathcal{M} \vDash \phi \Leftrightarrow \forall x \in X(\mathcal{M}, x \vDash \phi).$$

Let $\mathbb{X} = (X, \Omega)$ be a topological space, $\mathcal{C}$ be a class of spaces and ϕ be a formula. We say that a formula ϕ is valid in $\mathbb{X}$ (notation: $\mathbb{X} \vDash \phi$) if it is true in every model on this topological space, i.e.

$$\mathbb{X} \vDash \phi \Leftrightarrow \forall V\ (\mathbb{X}, V \vDash \phi).$$

We say that the formula ϕ is valid in $\mathcal{C}$ if it is *valid* in every space in $\mathcal{C}$.

Definition 2.1 The *logic* of a class of topological spaces $\mathcal{C}$ (denoted by $L(\mathcal{C})$) is the set of all formulas in the language $\mathcal{ML}_2$ that are valid in all spaces of the class $\mathcal{C}$.

Lemma 2.2 *Let* $\mathbb{X} = (X, \Omega)$ *be a topological space then* $\mathbb{X} \vDash AT_0$ *iff* $\mathbb{X}$ *is a*

T_0-space.

Proof. ($\Rightarrow$) The proof is by contradiction. Assume that $\mathbb{X} \vDash AT_0$ and let there be points x and y such that $x \neq y$ and $\forall U \in \Omega(x \in U \Leftrightarrow y \in U)$. Define a valuation V such that $V(p) = \{x\}$ and $V(q) = \{y\}$. Then

$$\mathbb{X}, V, x \vDash p \wedge [\neq]\neg p \wedge \langle\neq\rangle(q \wedge [\neq]\neg q)$$

and

$$\mathbb{X}, V, x \nvDash \Box\neg q \vee \langle\neq\rangle(q \wedge \Box\neg p).$$

This contradicts the fact that $\mathbb{X} \vDash AT_0$.

($\Leftarrow$)Assume that $\mathbb{X}$ is a T_0 space. Let

$$\mathbb{X}, V, x \vDash p \wedge [\neq]\neg p \wedge \langle\neq\rangle(q \wedge [\neq]\neg q).$$

Then there is a point y such that y is not equal to x and $V(q) = \{y\}$. Further, at least one of the points x and y is contained in a neighborhood that does not contain the other. That means $\mathbb{X}, V, x \vDash \Box\neg q$ or $\mathbb{X}, V, y \vDash \Box\neg p$ which proves our assertion. $\Box$

Kripke semantics is well-known (see [1]). We call the logic L complete with respect to the class of topological spaces $\mathcal{C}$ if $L(\mathcal{C}) = L$. The *logic* of a class of frames $\mathcal{C}$ (in notation $L(\mathcal{C})$) is the set of formulas that are valid in all frames from $\mathcal{C}$. For a single frame F, $L(F)$ stands for $L(\{F\})$. A logic L is called *Kripke complete* if there exists a class of frames $\mathcal{C}$, such that $L = L(\mathcal{C})$. A frame F is called an L-frame if $L \subseteq L(F)$.

Definition 2.3 Let $F = (W, R_1, ..., R_n)$ be a Kripke frame and S^* be the transitive and reflexive closure of the relation $S = (\bigcup_{i=0}^{n} R_i)$. For $x \in W$, $W^x \rightleftharpoons \{y \mid xS^*y\}$ (the set of all points reachable from the point x by relation S^*). The frame $F^x = (W^x, R_1|_{W^x}, ..., R_n|_{W^x})$ is called cone. If F is an L-frame, then the F^x is called the L-cone.

Lemma 2.4 *Let $F = (W, R, R_D)$ be an $S4D$-cone, then:*

$$F \vDash AT_0 \Longleftrightarrow \forall x, y \in W(xRy \wedge yRx \Longrightarrow xR_Dx \vee yR_Dy)$$

The axioms $T_\Box, 4_\Box, D_\Box, B_D, 4_D$ are Sahlqvist formulas. So we obtain the Kripke completeness for logic $S4D$ (see [1]). To prove the Kripke completeness of logic $S4DT_0$, we use lemma 2.3 and well-known canonical model construction (see [1], [2]).

3 Results

Theorem 3.1 *The logic $S4DT_0$ is complete with respect to topological T_0-spaces.*

For the proof we use the previous lemma and for each $S4DT_0$-cone we construct a special T_0-space. Next, we construct $p-morphism$ (see [3]) from spaces to corresponding frames and refer to the theorem on p-morphism.

Definition 3.2 A logic L has the $finite\,model\,property$ if $L = L(\mathcal{C})$, where $\mathcal{C}$ is a class of finite frames.

Definition 3.3 Let us consider a frame $F = (W, R_1, R_2)$ and an equivalence relation $\sim$ on W. A frame $F/{\sim} = (W/{\sim},\ R_1/{\sim},\ R_2/{\sim})$ is said to be a *minimal filtration* of F through $\sim$, if for $U_1, U_2 \in W/{\sim}$ and $i = 1, 2$

$$U_1 R_i/{\sim} U_2 \Leftrightarrow \exists u \in U_1\, \exists v \in U_2\, u R_i v$$

Definition 3.4 Let $M = (W, R_1,\ R_2, V)$ be a Kripke model, Φ a set of bimodal formulas closed under subformulas. For $x \in W$ let $\Phi(x) := \{A \in \Phi | M, x \vDash A\}$. Two worlds $x, y \in W$ are called Φ-*equivalent in* M (notation: $x \equiv_\Phi y$) if $\Phi(x) = \Phi(y)$.

We say that the equivalence $\sim$ agrees with a set Φ if $\sim \subseteq \equiv_\Phi$.

Lemma 3.5 *(cf. [5]) If a formula ϕ is satisfiable in model M over a frame F and the equivalence $\sim$ with a set of all subformulas of ϕ, then ϕ is satisfiable in $F/{\sim}$.*

A partition of the set W is a family of disjoint subsets of W whose union is W. If $\mathbb{A}$ and $\mathbb{B}$ are partitions of a set W and each element of $\mathbb{A}$ is a subset of one element from $\mathbb{B}$, then we say $\mathbb{A}$ is a refinement of $\mathbb{B}$. We denote by $\sim_\mathbb{A}$ the equivalence relation whose set of classes coincides with $\mathbb{A} : \mathbb{A} = W/{\sim_\mathbb{A}}$. We write $F_\mathbb{A}$ and $R_\mathbb{A}$ instead of $F/{\sim_\mathbb{A}}$ and $R/{\sim_\mathbb{A}}$.

Definition 3.6 A class of frames $\mathcal{C}$ *admits minimal filtration* if for each frame $F = (W, R, R_D) \in \mathcal{C}$ and for each finite partition $\mathbb{A}$ of W, there is a finite refinement $\mathbb{B}$ of $\mathbb{A}$, such that $F_\mathbb{B} \in \mathcal{C}$.

Lemma 3.7 *(cf. [7]) If $\mathcal{C}$ admits minimal filtration, then $L(\mathcal{C})$ has the finite model property.*

Theorem 3.8 *$S4DT_0$ has the finite model property.*

Proof.

Let there be an $S4DT_0$-cone $F = (W, R, R_D)$, in which the formula ϕ is satisfiable. We will show that there is a finite $S4DT_0$-frame in which ϕ is satisfiable. First we construct the minimal filtration of $M = (F, V)$ $(\exists x \in W\ (M,\ x \vDash \phi))$ ([1], [2]) via $\equiv_\Phi$, where Φ is the set of subformulas of ϕ, then we take the transitive closure of first relation and call the resulting frame as $M' = (F', V')$, where $F' = (W', R', R'_D)$. Note that each R'_D-irreflexive class consists of a single R_D-irreflexive point.

The resulting frame is not always an $S4DT_0$-frame, but always $S4D$-frame. Note that there is a finite number of points (equivalence classes) in F' and correspondingly a finite number of paths by the first relation from one R'_D-irreflexive point to another (paths such that no points are repeated). We consider only the classes entering into such paths and not being irreflexive with respect to the second relation. Let us somehow order these classes and we consider them one by one. Let a class y participate in m different paths. Somehow order these paths and we consider them in turn. Let y be visible from the class a and sees the class b. We devide points of class y into three classes. Note that we do not consider cases when a point of class y is visible from a point of class a and sees a point of class b. We skip such case and go to another path.

1. Points of class y that are visible from the class a will be denoted by N_1. Then N_1 is the first class.

1. Points of class y that see the class b are denoted by N_2. Then N_2 will be the second class.

3. The last class is $y \setminus (N_1 \cup N_2)$.

We continue this procedure for the next paths, but each time we consider classes obtained after partitioning instead of the classes considered. Note that this process is finite.

□

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Translating logics of uncertainty into two-layered modal fuzzy logics

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Abstract

This short paper provides a translation of the logic AX^M, introduced in [7] for reasoning about probabilities, into the logic $\mathrm{FP}(\text{Ł}_\triangle)$. The latter is a modal fuzzy logic with two syntactical layers: the lower one governed by classical logic and the upper one by Łukasiewicz logic extended with the projection connective $\triangle$. We also survey other logics for reasoning about uncertainty in the literature and hint at how they can benefit from a reformulation in terms of two-layered modal fuzzy logics.

1 Introduction

Logics for reasoning about uncertain events abound in the literature. Following Hamblin's [13], most authors conceived such logics as modal logics with a modality P standing for "is probable", or variants thereof (see e.g. [5, 7, 14]).

All such works display two important features:

(i) differently from usual modal logics, arbitrary nesting of modalities is not allowed,

(ii) despite dealing with intrinsically graded notions, such as probability, the semantics of these logics is essentially bivalent.

Indeed, these logics deal with statements of the form "φ is as probable as ψ" or "the probability of φ is greater or equal than 0.7".

An alternative approach in a many-valued setting, in particular in the framework of Mathematical Fuzzy Logic, takes sentences like "φ is probable"

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at face value, identifying its truth value with the probability of φ and, hence, shifting to the semantics the syntactical complexity of the previous approach. Such idea was proposed in [10,11] and later developed in Hájek's monograph [9] where the logic FP(Ł) was introduced. This logic has a special kind of syntax in which first one applies a modal operator P only on classical propositional formulas φ, in order to create atomic modal formulas $P\varphi$, and then combines the latter by the connectives of Łukasiewicz logic Ł.

This approach was later generalized to logics other than Łukasiewicz and uncertainty measures other than probabilities (see [8] for an overview).

An abstract study of these logics, known as *two-layered modal many-valued logics*, was proposed in [4]. Their distinctive feature is a two-layered syntax with: (i) non-modal formulas, (ii) atomic modal formulas obtained by applying the modality operator(s) only to non-modal ones, and (iii) complex modal formulas built from the atomic ones.

The exact relation between the formalism of logics of uncertainty spawning from Hamblin's seminal work and that of two-layered modal many-valued logics has not yet been described. We believe that the latter has a potential for a deeper understanding of the former, since the many-valued framework offers an amenable, well-studied, mathematical apparatus to deal in the semantics with the intended high syntactical complexity of logics of uncertainty. This paper reports on ongoing work towards this direction. After this introduction, in Section 2 we describe a faithful translation of the logic AX^M (defined in [7]) into FP(Ł) expanded with the projection connective $\triangle$. Section 3 ends the paper with some concluding remarks and hints at future research directions.

2 A translation of AX^M into $\mathrm{FP}(\text{Ł}_\triangle)$

Let us start by defining the language $\mathcal{L}^{\mathrm{QU}}$ of AX^M. First, the lower layer language is that of classical logic, i.e., *non-modal* formulas are those of classical propositional logic. Next, we introduce *basic inequality formulas* of the form $t \geq c$ where the *term* t is of the form $\sum_{i=1}^{n} a_i P(\varphi_i)$, φ_is are non-modal formulas, and c and a_i are constants for integers (a similar system presented in [12] uses real numbers instead, while the systems studied in [6] use rational coefficients). Using basic inequality formulas we define the *modal* formulas, via the following BNF grammar:

$$\psi ::= \bot \mid \top \mid t \geq c \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi \to \psi \mid \neg\psi.$$

Obvious abbreviations apply. In particular, we denote by $-t$ the term $\sum -a_i P(\varphi_i)$, if $t = \sum a_i P(\varphi_i)$, and we use $P(\varphi) \geq P(\psi)$ for the formula $P(\varphi) - P(\psi) \geq 0$, $t \leq c$ for the formula $-t \geq -c$, $t < c$ for the formula $\neg(t \geq c)$ and $t = c$ for the formula $(t \geq c) \wedge (t \leq c)$.

The logic AX^M over the language $\mathcal{L}^{\mathrm{QU}}$ is presented in [7] via an axiomatic system, which includes axioms of classical propositional logic, the rule of *modus ponens* for both modal and non-modal formulas, a set of axioms for manipulating linear inequalities (which we here omit), and the following:

(QU1)	$P(\phi) \geq 0$
(QU2)	$P(\top) = 1$
(QU3)	$P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) = P(\varphi)$
(QUGEN)	From $\varphi \leftrightarrow \psi$ infer $P(\varphi) = P(\psi)$.

Let us now briefly introduce FP(Ł$_\triangle$). We assume that the reader is familiar with the language and the standard semantics of Ł$_\triangle$, i.e. Łukasiewicz logic, extended with the connective $\triangle$, see e.g. [2] for more details. In the language of FP(Ł$_\triangle$) the non-modal formulas in the lower layer are classical, as in $\mathcal{L}^{\mathrm{QU}}$, but instead of basic inequality formulas we have simple atomic modal formulas of the form $P(\varphi)$ (for a classical formula φ). In the second layer, more complex modal formulas are built using the connectives of Ł$_\triangle$ as follows:

$$\psi ::= \bot_{Ł} \mid P(\varphi) \mid \triangle\psi \mid \psi \rightarrow_{Ł} \psi.$$

As it is well-known, other connectives usually considered for Łukasiewicz logic (i.e. $\wedge_{Ł}, \vee_{Ł}, \leftrightarrow_{Ł}, \oplus_{Ł}, \odot_{Ł}, \neg_{Ł}$) are definable from $\rightarrow_{Ł}$ and $\bot_{Ł}$. The axiom system for FP(Ł$_\triangle$) includes the axioms of classical propositional logic for non-modal formulas, axioms of Ł$_\triangle$ for modal formulas, *modus ponens* rules for both non-modal and modal formulas, and in addition:

(A1)	$P\varphi \rightarrow_{Ł} (P(\varphi \rightarrow \psi) \rightarrow_{Ł} P\psi)$
(A2)	$P\neg\varphi \leftrightarrow_{Ł} \neg_{Ł} P\varphi$
(A3)	$P(\varphi \vee \psi) \leftrightarrow_{Ł} [(P\varphi \rightarrow_{Ł} P(\varphi \wedge \psi)) \rightarrow_{Ł} P\psi]$
(NEC)	From φ infer $P\varphi$.

The semantic counterpart of FP(Ł$_\triangle$) are *probability Kripke frames*, that is, structures $\mathbf{F} = \langle W, \mathbf{2}, [0,1]_{Ł}, \mu\rangle$, where W is a set of possible worlds, $\mathbf{2}$ is the Boolean algebra of two elements, $[0,1]_{Ł}$ is the standard Łukasiewicz algebra and μ is a finitely additive probability measure. The idea is that non-modal (classical) formulas are evaluated in $\mathbf{2}$, μ is the interpretation of P, and modal formulas are interpreted in $[0,1]_{Ł}$. This is formally achieved by the notion of *Kripke model* over a probability Kripke frame $\mathbf{F}$, i.e. $\mathbf{M} = \langle \mathbf{F}, \langle e_w\rangle_{w\in W}\rangle$ where:

- e_w is a classical evaluation of non-modal formulas, for each $w \in W$,
- $[\varphi]_{\mathbf{M}} = \{w \mid e_w(\varphi) = 1\}$ is in the domain of μ, for each non-modal formula φ.

The evaluations e_ws are used to determine the truth value of non-modal formulas in a given world. The truth value of a formula $P(\varphi)$ in $\mathbf{M}$ is then defined as $||\varphi||_{\mathbf{M}} = \mu([\varphi]_{\mathbf{M}})$ and truth values of more complex formulas are defined using the corresponding operations in $[0,1]_{Ł}$. It follows from the general results in [4] that the logic FP(Ł$_\triangle$) is complete w.r.t. the semantics just introduced.

The semantics for AX^M is presented in slightly different terminology in [7], but can be equivalently reformulated in the style of that for FP(Ł$_\triangle$). Here we have structures of the kind $\mathbf{F} = \langle W, \mathbf{2}, \mathbf{2}, \mu\rangle$, where W is a set of possible worlds, $\mathbf{2}$ is the Boolean algebra of two elements, and μ is a finitely additive probability measure. A Kripke model over a probability Kripke frame $\mathbf{F}$ is defined for AX^M as $\mathbf{M} = \langle \mathbf{F}, \langle e_w\rangle_{w\in W}\rangle$ where e_w is a classical propositional evaluation for each $w \in W$. For basic inequality formulas, we let $||\sum a_i P(\gamma_i) \geq c||_{\mathbf{M}} = 1$ iff $\sum a_i \mu([\gamma_i]_{\mathbf{M}}) \geq c$ (abusing notation, we do not distinguish between constants for integers and the integers themselves). The interpretation $||\varphi||_{\mathbf{M}}$ of a complex formula φ is then obtained by the usual truth-functional extension. In [7] it is shown, essentially by reduction to linear programming problems, that the logic AX^M is sound and complete for the semantics just given and that its satisfiability problem is NP-complete.

Let us now discuss the translation. Let $t \geq c$ be a basic inequality formula in $\mathcal{L}^{\mathrm{QU}}$, where t stands for $\sum_{i=1}^n a_i P(\gamma_i)$, and consider the linear polynomial with integer coefficients $f(x_1, \dots, x_n) := \sum_{i=1}^n a_i x_i - c + 1$. By the McNaughton Theorem (see e.g. Lemma 2.1.21 in [1]) one can algorithmically build from f a formula φ of Ł over propositional variables $p_1, \dots, p_n$, such that for any standard evaluation e of Ł, letting $e(p_i) = v_i \in [0, 1]$, we have $e(\varphi) = \max\{0, \min\{1, f(v_1, \dots, v_n)\}\}$. Let us denote by $(t \geq c)^*$ the formula resulting from $\triangle\varphi$ by replacing each propositional variable p_i in φ by $P(\gamma_i)$.

The translation $*$ is then extended to complex formulas in AX^M by letting $\bot^* = \bot_{Ł}$ and $(\varphi \to \psi)^* = \varphi^* \to_{Ł} \psi^*$ (recall that both the classical and Łukasiewicz connectives are definable from implication and bottom).

Now we are ready to formulate the main result of our contribution (where Γ^* denotes the set resulting from applying $*$ to each formula in Γ).

Theorem 2.1 *For each $\Gamma \cup \{\varphi\}$ finite set of formulas of $\mathcal{L}^{\mathrm{QU}}$, we have:*

$$\Gamma \vdash_{\mathrm{AX}^M} \varphi \text{ if and only if } \Gamma^* \vdash_{\mathrm{FP}(Ł_\triangle)} \varphi^*.$$

Its proof is semantic in nature and uses the completeness theorem of both logics. Note however that if we would manage to obtain a syntactic proof of its right-to-left direction we would obtain an alternative proof of completeness of AX^M.

3 Conclusion

The translation presented above showcases the power of the many-valued semantics. Indeed, AX^M uses a complex syntax (with many constants for numbers) to express inequalities involving probabilities of events, while $\mathrm{FP}(Ł_\triangle)$ can directly express such inequalities thanks to its well-behaved many-valued semantics satisfying McNaughton Theorem. Moreover, this comes with a substantial simplification of the axiomatization of the logic since, unlike AX^M, $\mathrm{FP}(Ł_\triangle)$ does not need any explicit axioms to manipulate linear inequalities. Translations of other logics of uncertainty are likely to bring similar benefits.

Let us indicate some directions for further research. First, we will consider the system introduced in [14], which allows for modal formulas like $P_{\geq r}\varphi$ standing for “the probability of φ is at least r” where r is a constant for a rational number. The language is simpler than that of AX^M, but the axiomatization includes a quite involved rule. We believe that it can also be translated into $\mathrm{FP}(Ł_\triangle)$. The next step should focus on more expressive systems, such as the logic $\mathrm{AX}^{M,\times}$ [7] which is strictly more expressive than AX^M: basic inequality terms use arbitrary polynomials rather than just linear ones. In particular, it allows to express independence of events. Another expressive system introduced in [5] includes modalities of the form $P_{\geq r}$ and binary modalities expressing that a formula provides probabilistic confirmation or disconfirmation for another (this allows to express independence as well). We conjecture that both systems in [5] and [7] are interpretable into a two-layered modal logic, with classical logic in the lower layer and the logic PŁ$_{\triangle}$ (with both the Łukasiewicz and Product conjunction [2]) in the upper one, possibly extended with constants for rational numbers. As a possible further benefit we may be able to provide analytic calculi for logics of uncertainty in the literature, where so far little is known (see e.g. [15]). Indeed, we plan to extend hypersequent calculi for fuzzy logics, in particular the one for Łukasiewicz logic [16], to the setting of two-layered modal logics and then export them via the translations.

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Kripke modalities with truth-functional gaps

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Abstract

We introduce and study a family of three-valued Kripke modalities, in which the third truth value serves as an error code for undefined truth. We discuss several meaningful ways of error propagation by the modalities and present a few initial observations on the resulting three-valued modal logic.

Keywords: Three-valued logic, Kripke semantics, truth gap, error propagation.

1 Introduction

We study relational (Kripke-style) modalities in a three-valued setting, where the third value $*$ represents an error code for an undefined truth value. The classical Kripke semantics of modal logic assumes that each proposition in each possible world, as well as the accessibility relation between each pair of worlds, is assigned one of the two truth values. We relax this assumption, accommodating situations in which the truth of propositions or the accessibility between worlds may not be well defined.

Example 1.1 Consider the proposition: "Necessarily, most crows are black." This can be modeled as the ratio b_w/c_w being larger than .5 in all accessible worlds, where c_w denotes the number of crows in the world w and b_w the number of black crows in w. In worlds where $c_w = 0$, it may be reasonable to regard the proposition $b_w/c_w > .5$ as neither true nor false, but rather undefined, and assign to it the third truth value $*$.

The evaluation of modal propositions then depends on the intended meanings of modalities in the presence of the error value $*$. For example, the proposition "necessarily, most crows are black" may either be understood as true or false, depending on the contingency of the black crow ratio in those accessible worlds where it is well-defined (ignoring the crow-free worlds); or as neither true nor false (accounting for the fact that "most" is ill defined in crow-free worlds). In this paper we discuss several such systematic truth-valuation and error-propagation modes for modalities in gap-tolerant Kripke frames; our approach differs from known three-valued variants of modal logic such as [6].

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Example 1.2 Similarly there are cases where the accessibility between two possible worlds may be regarded as neither true nor false, but rather in some sense ill defined. Examples may include deontic alternative worlds that are neither expressly permitted nor prohibited; epistemic alternatives neither definitely admitted nor excluded by the agent; events outside the light cone (so neither in the absolute past nor the absolute future) of a reference event in relativistic temporal logic; etc.

Remark 1.3 While it might be useful to introduce different error codes for the undefined truth of modal propositions and accessibility relations, in this paper we only consider a single error code $*$ for both failures. We also restrict our attention to truth-functional propagation of the error code by the connectives of three-valued logic; more general cases are left for future work.

2 Three-valued connectives and quantifiers

Let $\mathcal{L}_3$ denote three-valued propositional logic with a functionally complete language. We will make use of the following unary and binary connectives of $\mathcal{L}_3$ (cf. [4], [3]):

x	$\sim$	$\downarrow$
0	1	0
1	0	1
$*$	$*$	0

x	y	$\to_{\mathrm{B}}$	$\to_{\mathrm{S}}$	$\to_{\mathrm{K}}$	$\to_{\mathrm{N}}$	$\to_{\mathrm{U}}$	$\wedge_{\mathrm{B}}$	$\wedge_{\mathrm{S}}$	$\wedge_{\mathrm{K}}$	$\wedge_{\mathrm{N}}$	$\wedge_{\mathrm{U}}$	$\equiv$
0	0	1	1	1	1	$*$	0	0	0	0	$*$	1
0	1	1	1	1	1	$*$	0	0	0	0	$*$	0
0	$*$	$*$	1	1	1	$*$	$*$	0	0	0	$*$	0
1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
1	$*$	$*$	0	$*$	$*$	$*$	$*$	1	$*$	$*$	$*$	0
$*$	0	$*$	0	$*$	1	$*$	$*$	0	0	0	$*$	0
$*$	1	$*$	1	1	1	$*$	$*$	1	$*$	0	$*$	0
$*$	$*$	$*$	$*$	$*$	1	$*$	$*$	$*$	$*$	0	$*$	1

Here, the *Bochvar* connectives $\to_{\mathrm{B}}, \wedge_{\mathrm{B}}$ regard $*$ as a fatal error; the *Sobociński* connectives $\to_{\mathrm{S}}, \wedge_{\mathrm{S}}$ regard $*$ as an ignorable error; and the *Kleene* (strong) connectives $\to_{\mathrm{K}}, \wedge_{\mathrm{K}}$ as an overridable error. The connective $\to_{\mathrm{N}}$ is the *Nelson* three-valued implication and $\wedge_{\mathrm{N}}$ the corresponding conjunction; and the connective $\to_{\mathrm{U}} = \wedge_{\mathrm{U}}$ will become useful in Section 4. Further families of three-valued connectives are definable by means of the listed ones, e.g., Bochvar's *external* connectives $\varphi \to_{\mathrm{E}} \psi \equiv_{\mathrm{df}} {\downarrow}\varphi \to_{\mathrm{B}} {\downarrow}\psi$ and $\varphi \wedge_{\mathrm{E}} \psi \equiv_{\mathrm{df}} {\downarrow}\varphi \wedge_{\mathrm{B}} {\downarrow}\psi$, which treat the error code $*$ as falsity.

Three-valued first-order models are defined as usual, with n-ary predicates interpreted by functions $D^n \to \{0, 1, *\}$, where D is the domain of the model. We will make use of the Bochvar, Sobociński, and Kleene three-valued quantifiers, which treat $*$ analogously as the corresponding connectives:

$$\|(\forall_{\mathrm{B}} x)\varphi\| = \begin{cases} 1 & \text{if } \|\varphi\|(a) = 1 \text{ for each } a \in D \\ * & \text{if } \|\varphi\|(a) = * \text{ for some } a \in D \\ 0 & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{S}}x)\varphi\| = \begin{cases} * & \text{if } \|\varphi\|(a) = * \text{ for each } a \in D \\ 0 & \text{if } \|\varphi\|(a) = 0 \text{ for some } a \in D \\ 1 & \text{otherwise} \end{cases}$$

$$\|(\forall_{\mathrm{K}}x)\varphi\| = \begin{cases} 1 & \text{if } \|\varphi\|(a) = 1 \text{ for each } a \in D \\ 0 & \text{if } \|\varphi\|(a) = 0 \text{ for some } a \in D \\ * & \text{otherwise} \end{cases}$$

and the dual existential quantifiers $(\exists_{\mathrm{X}}x)\varphi \equiv_{\mathrm{df}} \sim(\forall_{\mathrm{X}}x)\sim\varphi$ for $\mathrm{X} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$.

The entailment relation $\models_{\mathcal{L}_3^\forall}$ for the first-order three-valued logic $\mathcal{L}_3^\forall$ is defined in the standard manner, with only the truth value 1 designated (so $*$ is regarded as a truth-value gap rather than glut). The validity of various laws of $\mathcal{L}_3^\forall$ is readily verifiable (cf. [5]); for instance, the rule of generalization $\varphi \models_{\mathcal{L}_3^\forall} (\forall_{\mathrm{X}}x)\varphi$ holds for any $\mathrm{X} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$, while specification $(\forall_{\mathrm{X}}x)\varphi \models_{\mathcal{L}_3^\forall} \varphi$ only for $\mathrm{X} \in \{\mathrm{B}, \mathrm{K}\}$.

3 Three-valued Kripke models

Our aim is to expand the propositional language of $\mathcal{L}_3$ by three-valued Kripke modalities that propagate the error code $*$ in meaningful uniform ways. To this end, we need first to generalize Kripke models to the three values $\{0, 1, *\}$:

Definition 3.1 A *three-valued Kripke frame* is a structure $\mathbf{K} = (W, R)$, where $W \neq \emptyset$ and $R \colon W^2 \to \{0, 1, *\}$. A *three-valued Kripke model* over $\mathbf{K}$ is a pair $\mathbf{M} = (\mathbf{K}, e)$ with $e \colon W \times \mathrm{Var} \to \{0, 1, *\}$, where Var denotes the set of propositional variables.

Further on, let a three-valued Kripke model $\mathbf{M}$ be fixed. For (non-modal) $\mathcal{L}_3$-formulae φ, the truth value $\|\varphi\|_w \in \{0, 1, *\}$ of φ in $w \in W$ is given by the truth tables of three-valued propositional connectives. Depending on the intended modes of error-propagation, various three-valued Kripke modalities can be introduced in $\mathbf{M}$. A rather general definition schema parameterizes them by the three-valued connective and quantifier employed for the accessibility-relative quantification over worlds:

Definition 3.2 The three-valued Kripke modalities $\Box_{\mathrm{XY}}, \Diamond_{\mathrm{XY}}$ are defined by the following Tarski conditions in a three-valued Kripke model $\mathbf{M}$:

$$\|\Box_{\mathrm{XY}}\,\varphi\|_w = (\forall_{\mathrm{X}}w')(Rww' \to_{\mathrm{Y}} \|\varphi\|_{w'})$$
$$\|\Diamond_{\mathrm{XY}}\,\varphi\|_w = (\exists_{\mathrm{X}}w')(Rww' \wedge_{\mathrm{Y}} \|\varphi\|_{w'}),$$

where $\mathrm{X} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}, \ldots\}$ and $\mathrm{Y} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}, \mathrm{M}, \mathrm{N}, \mathrm{U}, \mathrm{E}, \ldots\}$, the dots standing for further possible families of definable quantifiers and connectives of $\mathcal{L}_3^\forall$.

The modal logic $\mathcal{L}_3^\Box$ of three-valued Kripke models (in the language of $\mathcal{L}_3$ expanded by all modalities $\Box_{\mathrm{XY}}, \Diamond_{\mathrm{XY}}$) is defined in the standard manner; again, only the truth value 1 is regarded as designated.

Since three-valued Kripke models can be identified with three-valued first-order models for the language consisting of a binary predicate R and unary predicates P_i for each propositional letter p_i, the logic $\mathcal{L}_3^{\Box}$ can be faithfully interpreted in $\mathcal{L}_3^{\forall}$ by a syntactic translation (cf. [7, Ch. 8]):

Definition 3.3 For each $\mathcal{L}_3^{\Box}$-formula φ and object variable x define the $\mathcal{L}_3^{\forall}$-formula $\varphi^\sharp(x)$ as follows:

- $p_i^\sharp(x)$ is $P_i x$, for each propositional letter p_i
- $\sharp$ commutes with $\mathcal{L}_3$-connectives, so $(\varphi \to_{\mathrm{B}} \psi)^\sharp(x)$ is $\varphi^\sharp(x) \to_{\mathrm{B}} \psi^\sharp(x)$ etc.
- $(\Box_{\mathrm{XY}}\, \varphi)^\sharp(x)$ is $(\forall_{\mathrm{X}} y)(Rxy \to_{\mathrm{Y}} \varphi^\sharp(y))$, for y not free in $\varphi^\sharp(x)$
- $(\Diamond_{\mathrm{XY}}\, \varphi)^\sharp(x)$ is $(\exists_{\mathrm{X}} y)(Rxy \wedge_{\mathrm{Y}} \varphi^\sharp(y))$, for y not free in $\varphi^\sharp(x)$

Lemma 3.4 *For an $\mathcal{L}_3^{\Box}$-model $\mathbf{M} = ((W,R),e)$ let $\mathbf{M}^\sharp = (W, R^\sharp, P_1^\sharp, P_2^\sharp, \dots)$ be an $\mathcal{L}_3^{\forall}$-model such that $R^\sharp = R$ and $P_i^\sharp w = e(w, p_i)$ for each i and $w \in W$.*

Then $\|\varphi\|_w = \|\varphi^\sharp(x)\|_{\mathbf{M}^\sharp, v}$ for any $\mathbf{M}^\sharp$-valuation v such that $v(x) = w$. Moreover, $\sharp\colon \mathbf{M} \mapsto \mathbf{M}^\sharp$ is a one-to-one correspondence between $\mathcal{L}_3^{\Box}$-models and $\mathcal{L}_3^{\forall}$-models for the language $R, P_1, P_2, \dots$

Corollary 3.5 $\Gamma \models_{\mathcal{L}_3^{\Box}} \varphi$ *iff* $\Gamma^\sharp \models_{\mathcal{L}_3^{\forall}} \varphi^\sharp$.

The proofs are routine. Since $\mathcal{L}_3^{\forall}$ is recursively enumerable (axiomatizable), Corollary 3.5 provides a syntactic method of generating the tautologies of $\mathcal{L}_3^{\Box}$ (although not an axiomatization in the language of $\mathcal{L}_3^{\Box}$). Moreover, since Lemma 3.4 and Corollary 3.5 can be straightforwardly generalized for classes of $\mathcal{L}_3^{\Box}$-models and the corresponding $\mathcal{L}_3^{\forall}$-models, the recursive enumerability applies as well to classes of $\mathcal{L}_3^{\Box}$-models with the accessibility relation restricted by an $\mathcal{L}_3^{\forall}$-definable condition (such as three-valued reflexivity, transitivity, etc.).

4 Prominent three-valued modalities

Of the large number of possible three-valued modalities $\Box_{\mathrm{XY}}, \Diamond_{\mathrm{XY}}$ introduced by Definition 3.2, only a few are well-behaved and conforming with the motivations of Section 1. Essentially, there are two ways of treating $*$-accessible worlds: either screening them off (i.e., regarding them as inaccessible), or taking them into account in some suitable manner.

Among the most meaningful modalities that screen off $*$-accessible worlds are $\Box_{\mathrm{BN}}, \Box_{\mathrm{KN}}, \Box_{\mathrm{SU}}, \Box_{\mathrm{BE}}$ and their duals $\Diamond_{\mathrm{BN}}, \Diamond_{\mathrm{KN}}, \Diamond_{\mathrm{SU}}, \Diamond_{\mathrm{BE}}$. Of these, $\Box_{\mathrm{BN}}$ and $\Diamond_{\mathrm{BN}}$ exhibit a Bochvar-style behavior, as

$$\|\Box_{\mathrm{BN}}\, \varphi\|_w = \begin{cases} 1 & \text{if } \|\varphi\|_{w'} = 1 \text{ for each 1-accessible world } w' \\ * & \text{if } \|\varphi\|_{w'} = 1 \text{ for some 1-accessible world } w' \\ 0 & \text{otherwise} \end{cases}$$

and dually for $\Diamond_{\mathrm{BN}}$ (cf. $\forall_{\mathrm{B}}$ and $\exists_{\mathrm{B}}$ in Sect. 2). Similarly, $\Box_{\mathrm{KN}}$ and $\Diamond_{\mathrm{KN}}$ behave Kleene-style, $\Box_{\mathrm{SU}}, \Diamond_{\mathrm{SU}}$ Sobociński-style, and $\Box_{\mathrm{BE}}, \Diamond_{\mathrm{BE}}$ Bochvar-external style.

Of the modalities that take $*$-accessible worlds into account, some of the reasonable ones are $\Box_{\mathrm{BK}}, \Diamond_{\mathrm{BK}}$ (Bochvar-like) and $\Box_{\mathrm{KK}}, \Diamond_{\mathrm{KK}}$ (Kleene-like). Further modalities $\Box_{\mathrm{XY}}, \Diamond_{\mathrm{XY}}$ may be suitable for specific purposes: just like with $\mathcal{L}_3$-connectives and $\mathcal{L}_3^{\forall}$-quantifiers, the rich repertory of $\mathcal{L}_3^{\Box}$-modalities makes it possible for the user to choose error-propagation modes that suit the intended application.

Remark 4.1 Note that using asymmetric connectives (such as $\rightarrow_{\mathrm{N}}, \wedge_{\mathrm{N}}$ or $\wedge_{\mathrm{U}}$) in Definition 3.2 partly remedies the conflation of undefined propositions and undefined accessibility by a single error code $*$ (Remark 1.3), as they treat $Rww' = *$ and $\|\varphi\|_{w'} = *$ differently. (Still, e.g., the fact that $\|\Box_{\mathrm{SU}}\,\varphi\| = *$ gives no indication as to which of the two kinds of error has occurred.)

5 Properties of the three-valued modalities

By Corollary 3.5, the laws of $\mathcal{L}_3^{\Box}$ are readily derivable from those of $\mathcal{L}_3^{\forall}$. Due to space limitations, we only give a few examples of $\mathcal{L}_3^{\Box}$-valid rules.

Proposition 5.1

(i) $\models_{\mathcal{L}_3^{\Box}} \Diamond_{\mathrm{XY}}\,\varphi \equiv {\sim}\,\Box_{\mathrm{XY}}\,{\sim}\varphi$
whenever $\models_{\mathcal{L}_3} {\sim}(p \rightarrow_{\mathrm{Y}} q) \equiv (p \wedge_{\mathrm{Y}} {\sim}q)$ *and* $\models_{\mathcal{L}_3^{\forall}} (\exists_{\mathrm{X}} x)Px \equiv {\sim}(\forall_{\mathrm{X}} x){\sim}Px$
(in particular, if $\mathrm{X} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$ *and* $\mathrm{Y} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}, \mathrm{N}, \mathrm{U}, \mathrm{E}\}$*).*

(ii) $\varphi \models_{\mathcal{L}_3^{\Box}} \Box_{\mathrm{XY}}\,\varphi$
whenever $\mathcal{L}_3^{\forall}$ *validates generalization for* $\forall_{\mathrm{X}}$ *and weakening for* $\rightarrow_{\mathrm{Y}}$
(so for any $\mathrm{X} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}\}$ *and* $\mathrm{Y} \in \{\mathrm{S}, \mathrm{K}, \mathrm{N}, \mathrm{E}\}$*, but not for* $\mathrm{Y} \in \{\mathrm{B}, \mathrm{U}\}$*).*

(iii) $\Box_{\mathrm{XY}}(\varphi \rightarrow_{\mathrm{Z}} \psi) \models_{\mathcal{L}_3^{\Box}} \Box_{\mathrm{XY}}\,\varphi \rightarrow_{\mathrm{Z}} \Box_{\mathrm{XY}}\,\psi$
whenever $\forall_{\mathrm{X}}$ *and* $\rightarrow_{\mathrm{Y}}$ *distribute over* $\rightarrow_{\mathrm{Z}}$ *in* $\mathcal{L}_3^{\forall}$
(so, e.g., when $\mathrm{X} \in \{\mathrm{B}, \mathrm{K}\}$*,* $\mathrm{Y} \in \{\mathrm{K}, \mathrm{N}\}$*, and* $\mathrm{Z} \in \{\mathrm{B}, \mathrm{E}\}$*).*

(iv) $\|\Box_{\mathrm{XY}}\,\varphi\|_w = 1$ *implies* $\|\varphi\|_w = 1$ *in* $\mathbf{M} = ((W, R), e)$ *if* $Rww = 1$
(i.e., the rule $\Box_{\mathrm{XY}}\,\varphi \models \varphi$ *is valid in reflexive* $\mathcal{L}_3^{\Box}$*-frames)*
for any $\mathrm{X} \in \{\mathrm{B}, \mathrm{K}\}$ *(but not* $\mathrm{X} = \mathrm{S}$*) and* $\mathrm{Y} \in \{\mathrm{B}, \mathrm{S}, \mathrm{K}, \mathrm{N}, \mathrm{U}, \mathrm{E}\}$.

Various further properties of $\mathcal{L}_3^{\Box}$ can be derived from those of $\mathcal{L}_3^{\forall}$ by Lemma 3.4. The investigation of logical and metamathematical features of $\mathcal{L}_3^{\Box}$ is a work in progress, and part of a broader study of many-valued (fuzzy) logics with truth-functionally propagated truth-value gaps (see, e.g., [3], [2], [5], [1]).

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Topological Possibility Frames

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Abstract

Possibility semantics for modal logic is a generalization of possible world semantics, based on partially ordered sets of region-like "possibilities" instead of only point-like "worlds." Here we adopt a topological perspective on possibility semantics. Just as one can view ordinary general frames topologically, one can also view general possibility frames topologically. The result is the notion of *topological possibility frames* introduced here. The advantage of topological possibility frames over the topological versions of ordinary general frames is that only the former enable a choice-free duality theory for modal algebras. This yields the modal version of the choice-free topological duality theory for Boolean algebras recently proposed by the authors in [2].

Keywords: modal logic, topology, duality theory, choice free, possibility semantics

1 Introduction

In possible world semantics, a Boolean algebra (BA) of propositions is realized as a field of sets. In the generalization known as *possibility semantics* [5,4,1], a BA of propositions is realized as the regular open algebra of a poset or subalgebra thereof. By the regular open algebra of a poset $(S, \sqsubseteq)$, we mean the BA of regular open sets of the corresponding Alexandroff space whose open sets are the $\sqsubseteq$-upsets of $(S, \sqsubseteq)$, so $\mathrm{int}_{\sqsubseteq} U = \{x \in S \mid \forall x' \sqsupseteq x : x' \in U\}$ and $\mathrm{cl}_{\sqsubseteq} U = \{x \in S \mid \exists x' \sqsupseteq x : x' \in U\}$. Then U is regular open iff $U = \mathrm{int}_{\sqsubseteq}\mathrm{cl}_{\sqsubseteq} U = \{x \in S \mid \forall x' \sqsupseteq x\ \exists x'' \sqsupseteq x' : x'' \in U\}$. As observed by Tarski and Stone, any collection P of regular open subsets of a space such that P is closed under intersection and the operation $\neg$ given by $\neg U = \mathrm{int}_{\sqsubseteq}(S \setminus U)$ forms a BA under these operations, and if a family $\{U_i \mid i \in I\}$ has a join in this BA, then it is given by $\bigvee\{U_i \mid i \in I\} = \mathrm{int}_{\sqsubseteq}\mathrm{cl}_{\sqsubseteq} \bigcup\{U_i \mid i \in I\}$. Possibility semantics then adds a binary relation R that induces an operation $\Box_R$ on the BA as usual by $\Box_R U = \{x \in S \mid \forall y : xRy \Rightarrow y \in U\}$.

Here we push the topological view of possibility semantics further by using the distinguished collection P of regular open sets of $(S, \sqsubseteq)$ to generate a topology on S, leading to a new notion of *topological possibility frames*.

2 World Frames and Possibility Frames

Let us first fix terminology and notation while reviewing the basic notions concerning world frames and possibility frames.

Definition 2.1

(i) A *general world frame* is a triple $\mathfrak{F} = (W, R, P)$ where W is a nonempty set, R is a binary relation on W, and P is a field of subsets of W closed under the operation $\Box_R$ defined by $\Box_R U = \{w \in W \mid \forall v \in W : wRv \Rightarrow v \in U\}$.

(ii) A general world frame is *descriptive* if it satisfies, for all $w, v \in W$:
  (a) *differentiation*: $w = v$ iff for all $U \in P$, $w \in U$ iff $v \in U$;
  (b) *R-tightness*: if for all $U \in P$, $w \in \Box_R U$ implies $v \in U$, then wRv;
  (c) *ultrafilter realization*: every ultrafilter in P is $P(w)$ for some $w \in W$, where $P(w) = \{U \in P \mid w \in U\}$.

The following is well known, with part (ii) proved in [3].

Theorem 2.2

(i) *For any general world frame $\mathfrak{F} = (W, R, P)$, P is a BA under intersection and set-theoretic complement that becomes a modal algebra $\mathfrak{F}^*$ with the multiplicative operation $\Box_R$.*

(ii) *Every modal algebra is isomorphic to $\mathfrak{F}^*$ for a descriptive general world frame $\mathfrak{F}$.*

The analogous notions for possibility semantics from [4] are the following.

Definition 2.3

(i) A *general possibility frame* is a quadruple $\mathcal{F} = (S, \sqsubseteq, R, P)$ where $(S, \sqsubseteq)$ is a poset, R is a binary relation on S, and P is a collection of regular open subsets of $(S, \sqsubseteq)$ closed under intersection and $\neg$ and $\Box_R$ from § 1.

(ii) A general possibility frame is *filter-descriptive* if it satisfies, for all $x, y \in S$:
  (a) *$\sqsubseteq$-tightness*: if for all $U \in P$, $x \in U$ implies $y \in U$, then $x \sqsubseteq y$;
  (b) *R-tightness*: if for all $U \in P$, $x \in \Box_R U$ implies $y \in U$, then xRy;
  (c) *filter realization*: every filter in P is $P(x)$ for some $x \in W$, where $P(x) = \{U \in P \mid x \in U\}$.

A key difference between Theorem 2.2.ii and the following theorem from [4] is that the former requires a nonconstructive choice principle, equivalent to the Boolean Prime Ideal Theorem, whereas the latter is provable in ZF set theory.

Theorem 2.4

(i) *For any possibility frame $\mathcal{F} = (W, R, P)$, P is a Boolean algebra under intersection and the operation $\neg$ that becomes a modal algebra $\mathcal{F}^*$ with the multiplicative operation $\Box_R$.*

(ii) *(ZF) Every modal algebra is isomorphic to $\mathcal{F}^*$ for a filter-descriptive possibility frame $\mathcal{F}$.*

3 Topological Frames

The following notion is not standard, but it appears implicitly in [7].

Definition 3.1 A *topological world frame* is a triple $\mathfrak{T} = (W, R, \tau)$ where (W, τ) is a zero-dimensional topological space and R is a binary relation on W such that $\Box_R$ sends clopens to clopens.

Definition 3.2 Given a topological world frame $\mathfrak{T} = (W, R, \tau)$, define its *associated general world frame* $G(\mathfrak{T}) = (W, R, P)$ where P is the set of clopens of (W, τ). Given a general world frame $\mathfrak{F} = (W, R, P)$, define its *associated topological world frame* $T(\mathfrak{F}) = (W, R, \tau)$ where τ is the topology generated by taking the elements of P as basic opens.

The notion of a topological world frame is a generalization of the standard notion of a *modal space*, which we call a 'modal Stone space'.

Definition 3.3 A *modal Stone space* is a topological world frame $\mathfrak{T} = (W, R, \tau)$ in which (W, τ) is compact and Hausdorff, and R is point-closed, i.e., for every $w \in W$, $R(w)$ is a closed subset of (W, τ).

Modal Stone spaces are the topological versions of descriptive general frames. This connection is decomposed in the following proposition from [7].

Proposition 3.4 *For any topological world frame $\mathfrak{T}$:* (i) $T(G(\mathfrak{T}))$ *is isomorphic to $\mathfrak{T}$;* (ii) *$\mathfrak{T}$ is compact iff $G(\mathfrak{T})$ satisfies ultrafilter realization;* (iii) *$\mathfrak{T}$ is Hausdorff iff $G(\mathfrak{T})$ is differentiated;* (iv) *$\mathfrak{T}$ is Hausdorff and R is point-closed iff $G(\mathfrak{T})$ is differentiated and R-tight.*

For any general world frame $\mathfrak{F}$: (v) *if $\mathfrak{F}$ satisfies ultrafilter realization, then $\mathfrak{F}$ is isomorphic to $G(T(\mathfrak{F}))$;* (vi) *$\mathfrak{F}$ satisfies ultrafilter realization iff $T(\mathfrak{F})$ is compact;* (vii) *$\mathfrak{F}$ is differentiated iff $T(\mathfrak{F})$ is Hausdorff;* (viii) *$\mathfrak{F}$ is differentiated and R-tight iff $T(\mathfrak{F})$ is Hausdorff and R is point-closed.*

For the analogous possibility semantic notions, we need a new order-topological notion. As an analogy, recall other order-topological dualities: Priestley duality and Esakia duality uses clopen $\sqsubseteq$-upsets.

Definition 3.5 Let $(S, \sqsubseteq)$ be a poset and (S, τ) a space. A set $U \subseteq S$ is *neg-closed* if $\neg U$ is open in (S, τ), with $\neg$ defined from $(S, \sqsubseteq)$ as in § 1. A set $U \subseteq S$ is *neg-clopen* if U is both open in (S, τ) and neg-closed. Let $\mathsf{NegClop}\mathcal{RO}(S, \sqsubseteq, \tau)$ be the set of all $U \subseteq S$ that are neg-clopen in (S, τ) and regular open in $(S, \sqsubseteq)$.

Definition 3.6 A *topological possibility frame* is a quadruple $\mathcal{T} = (S, \sqsubseteq, R, \tau)$ such that $(S, \sqsubseteq)$ is a poset, (S, τ) is a topological space such that $\mathsf{NegClop}\mathcal{RO}(S, \sqsubseteq, \tau)$ is closed under intersection and forms a basis, and R is a binary relation on S such that $\Box_R$ sends elements of $\mathsf{NegClop}\mathcal{RO}(S, \sqsubseteq, \tau)$ to elements of $\mathsf{NegClop}\mathcal{RO}(S, \sqsubseteq, \tau)$. Let $\mathsf{NegClop}\mathcal{RO}(\mathcal{T}) := \mathsf{NegClop}\mathcal{RO}(S, \sqsubseteq, \tau)$.

Definition 3.7 Given a topological possibility frame $\mathcal{T} = (S, \sqsubseteq, R, \tau)$, define its *associated general possibility frame* $G(\mathcal{T}) = (S, \sqsubseteq, R, P)$ where $P = \mathsf{NegClop}\mathcal{RO}(\mathcal{T})$. Given a general possibility frame $\mathcal{F} = (S, \sqsubseteq, R, P)$, define its *associated topological possibility frame* $T(\mathcal{F}) = (S, \sqsubseteq, R, \tau)$ where τ is the topology generated by taking the elements of P as basic opens.

The possibility semantic analogues of modal Stone spaces are the following modal versions of the UV-spaces of [2]. Here 'UV' stands for *upper Vietoris*, because the relevant topological spaces also arise as the hyperspace of nonempty closed subsets of a Stone space endowed with the upper Vietoris topology.

Definition 3.8 A *modal UV-space* is a topological possibility frame $\mathcal{T} = (S, \sqsubseteq, R, \tau)$ in which (S, τ) is a T_0 space and $\sqsubseteq$-tightness, R-tightness, and filter realization from Definition 2.3.ii hold for $P = \mathsf{NegClop}\mathcal{RO}(\mathcal{T})$.

Table 1 summarizes the relations between the frame classes above.

	world semantics	**possibility semantics**
gen. frames	gen. world frame	gen. poss. frames
gen. frames dual to algebras	desc. gen. world frame	filter-desc. gen. poss. frame
top. frames	top. world frames	top. poss. frames
top. frames dual to algebras	modal Stone space	modal UV-space

Table 1

Theorem 3.9

(i) *For any topological possibility frame $\mathcal{T} = (S, \sqsubseteq, R, \tau)$, $\mathsf{NegClop}\mathcal{RO}(\mathcal{T})$ is a BA under intersection and the operation $\neg$ that becomes a modal algebra $\mathcal{T}^*$ with the multiplicative operation $\Box_R$.*

(ii) *(ZF) Every modal algebra is isomorphic to $\mathcal{T}^*$ for a modal UV-space $\mathcal{T}$.*

We will sketch the proof of Theorem 3.9.ii, which is choice free in contrast to Theorem 2.2.ii. The construction of the modal UV-space $\mathcal{T}$ is as follows.

Definition 3.10 For any modal algebra $\mathbb{A}$, let $UV(\mathbb{A})$ be the topological possibility frame $(S, \sqsubseteq, R, \tau)$ where S is the set $\mathrm{PropFilt}(\mathbb{A})$ of proper filters of $\mathbb{A}$, $\sqsubseteq$ is $\subseteq$, R is defined by FRF' iff for all $a \in \mathbb{A}$, $\Box a \in F$ implies $a \in F'$, and τ is the topology generated by taking as basic opens the sets $\widehat{a} = \{F \in \mathrm{PropFilt}(\mathbb{A}) \mid a \in F\}$ for each $a \in \mathbb{A}$.

Proposition 3.11 *For any modal algebra $\mathbb{A}$, $UV(\mathbb{A})$ is a modal UV-space.*

To prove that the modal algebra $\mathsf{NegClop}\mathcal{RO}(UV(\mathbb{A}))$ is isomorphic to $\mathbb{A}$, we need the following lemma, which is an analogue of Theorem 1.9.3 in [3].

Lemma 3.12 *Let $\mathcal{T} = (S, \sqsubseteq, R, \tau)$ be a modal UV-space. Suppose that $S = \bigvee\{U_i \mid i \in I\}$ for $U_i \in \mathsf{NegClop}\mathcal{RO}(\mathcal{T})$, where $\bigvee$ is the join in $\mathsf{NegClop}\mathcal{RO}(\mathcal{T})$. Then $S = \bigvee\{U_i \mid i \in I_0\}$ for some finite $I_0 \subseteq I$.*

We also need the following general topological fact.

Lemma 3.13 *For any space X, $V \subseteq X$, and open $U \subseteq X$, if $U \cap V = \varnothing$, then $U \cap \mathrm{int}(\mathrm{cl}(V)) = \varnothing$.*

The key fact for the proof of Theorem 3.9.ii is the following, for then the map $a \mapsto \widehat{a}$ is an isomorphism from $\mathbb{A}$ to $\mathsf{NegClop}\mathcal{RO}(UV(\mathbb{A}))$ ordered by $\subseteq$.

Proposition 3.14 $\mathsf{NegClop}\mathcal{RO}(UV(\mathbb{A})) = \{\widehat{a} \mid a \in \mathbb{A}\}$.

Proof. The right-to-left inclusion is easy to check. For the left-to-right, suppose $U \in \mathsf{NegClop}\mathcal{RO}(UV(\mathbb{A}))$. Since U is open, $U = \bigcup\{\widehat{a} \mid \widehat{a} \subseteq U\}$, and since U is neg-closed, $\neg U = \bigcup\{\widehat{b} \mid \widehat{b} \subseteq \neg U\}$. Then since $U, \neg U$ are regular open w.r.t $\sqsubseteq$, $U = \bigvee\{\widehat{a} \mid \widehat{a} \subseteq U\}$ and $\neg U = \bigvee\{\widehat{b} \mid \widehat{b} \subseteq \neg U\}$. Since $U \vee \neg U = S$, we have $S = \bigvee\{\widehat{a} \mid \widehat{a} \subseteq U\} \vee \bigvee\{\widehat{b} \mid \widehat{b} \subseteq \neg U\}$. It follows by Lemma 3.12 that $S = \widehat{a_1} \vee \cdots \vee \widehat{a_n} \vee \widehat{b_1} \vee \cdots \vee \widehat{b_m}$ where $\widehat{a_i} \subseteq U$ and $\widehat{b_i} \subseteq \neg U$. Since $U \cap \neg U = \varnothing$ and $\widehat{b_i} \subseteq \neg U$, we have $U \cap \widehat{b_i} = \varnothing$. Hence $U \cap (\widehat{b_1} \cup \cdots \cup \widehat{b_m}) = \varnothing$, which by Lemma 3.13 implies $U \cap (\widehat{b_1} \vee \cdots \vee \widehat{b_m}) = \varnothing$. From $S = \widehat{a_1} \vee \cdots \vee \widehat{a_n} \vee \widehat{b_1} \vee \cdots \vee \widehat{b_m}$ and the fact that the meet $\wedge$ in $\mathsf{NegClop}\mathcal{RO}(UV(\mathbb{A}))$ is intersection, we have:

$$\begin{aligned} U &= U \wedge ((\widehat{a_1} \vee \cdots \vee \widehat{a_n}) \vee (\widehat{b_1} \vee \cdots \vee \widehat{b_m})) \\ &= (U \wedge (\widehat{a_1} \vee \cdots \vee \widehat{a_n})) \vee (U \wedge (\widehat{b_1} \vee \cdots \vee \widehat{b_m})) \\ &= U \wedge (\widehat{a_1} \vee \cdots \vee \widehat{a_n}), \end{aligned}$$

where the last equation uses that $U \cap (\widehat{b_1} \vee \cdots \vee \widehat{b_m}) = \varnothing$. Since $\wedge$ is intersection, $U = U \wedge (\widehat{a_1} \vee \cdots \vee \widehat{a_n})$ implies $U \subseteq \widehat{a_1} \vee \cdots \vee \widehat{a_n}$. From above we have $\widehat{a_1} \cup \cdots \cup \widehat{a_n} \subseteq U$, which with U being regular open w.r.t $\sqsubseteq$ implies $\widehat{a_1} \vee \cdots \vee \widehat{a_n} \subseteq U$. Thus, $U = \widehat{a_1} \vee \cdots \vee \widehat{a_n}$, which implies $U = \widehat{a_1 \vee \cdots \vee a_n}$ as shown in [4,2].□

Remark 3.15 In [2], it is shown that $\{\widehat{a} \mid a \in \mathbb{A}\} = \mathsf{CO}\mathcal{RO}(UV(\mathbb{A}))$, where $\mathsf{CO}\mathcal{RO}(\mathcal{T})$ is the collection of sets that are *compact open* in (S, τ) and regular open in $(S, \sqsubseteq)$. For modal UV-spaces, $\mathsf{NegClop}\mathcal{RO}(\mathcal{T}) = \mathsf{CO}\mathcal{RO}(\mathcal{T})$, but this equality does not hold for arbitrary topological possibility frames.

Like the standard topological duality for modal algebras using modal Stone spaces, the topological duality for modal algebras using modal UV-spaces allows one to bring topological intuitions to bear on problems of modal logic, but now without the need for nonconstructive choice principles. The question of what is achievable without choice has been of considerable interest in a wider topological context [6], and we find it of interest in a modal context too.

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Interpolation Results for Default Logic Over Modal Logic

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Abstract

Interpolation is an important meta property of a logic. We study interpolation results for a prioritized variant of Default Logic built over the Modal Logic KDA, the normal modal logic K extended with the axiom D for seriality and the universal modality A.

Keywords: Interpolation, Modal Logic, Default Logic

1 Introduction

Default Logics are among the best-known Nonmonotonic Logics. Their origins can be traced back to Reiter's seminal paper 'A Logic for Default Reasoning' [15]. Since then many variants and *addenda* have been proposed to Reiter's original ideas [2]. Default Logics have been thoroughly studied from the point of view of nonmonotonic consequence relations. However, with some exceptions, in particular [1], little attention has been paid to the study of interpolation results for them. And even less to the study of interpolation results for Default Logics built over Modal Logics.

The combination of Default Logics and Modal Logics is particularly interesting when reasoning about description and prescription. This kind of reasoning is prevalent in diverse areas such as Artificial Intelligence, Software Engineering, Legal Argumentation, etc.cas Typical descriptive statements refer to basic properties of a domain or scenario. Prescriptive statements are regulatory statements characterising ideal behaviours or situations. One main difficulty in dealing with these kinds of statements occurs when the information regarding the domains changes in a way such that the original prescriptions are overridden; or when prescriptions from different sources contradict each other. The tools of Deontic Logics allow for a distinction between descriptive and prescriptive statements, and the violation and fulfilment of prescriptions; and the tools of Default Logics make it possible to effectively reason about overriding prescriptions, and contradictory descriptions or prescriptions. For these reasons, we develop a Default Logic over Deontic Logic, called $\mathscr{D}\mathsf{KDA}$.

We resort to Deontic Logics as they provide a strong logical basis for the study of prescriptions (norms). Deontic Logics originate from the pioneer work of von Wright [16] and have been largely defined as particular Modal Logics [6,4]. The most famous is *Standard Deontic Logic* (SDL), i.e., the normal modal logic K extended with the axiom D for seriality [8,3].

In this short paper, we set out to study interpolation results for $\mathscr{D}\mathsf{KDA}$. Interpolation is an important meta property of a logic [10]. First formulated by Craig in [9], in one of its forms, the property states that if $\Phi \vdash \varphi \supset \psi$, then, there is θ s.t. $\Phi \vdash \varphi \supset \theta$, $\Phi \vdash \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$; where $\vdash$ is syntactical consequence in FOL, $\Phi \cup \{\varphi, \theta, \psi\}$ is a set of FOL formulas, and $\mathscr{L}$ is the set of non-logical symbols of a formula. Interpolation results for Modal Logics can be found in [12,5]. As a property, interpolation is worth studying for has direct applications in the area of theorem proving, the analysis and verification of programs, and in synthesis, e.g., in the generation of invariants. Interpolation also has an application in the structuring of specifications, e.g., in [14] it is proven that a form of interpolation is needed in order to compose specifications (the so-called *Modularization Theorem*). Having in mind similar application areas for $\mathscr{D}\mathsf{KDA}$ it seems natural to try and reproduce some interpolation results for this Default Logic. However, because of its non-monotonic nature and the composite structure of its premiss sets, one of the main challenges regarding interpolation results seems to be finding an adequate notion of interpolation. We will discuss some alternatives, taking advantage of interpolation properties of the underlying Modal Logic.

2 The Modal Logic KDA

Let $\mathscr{P}$ be a denumerable set of *proposition symbols*, the set $\mathscr{F}$ of wffs of KDA is determined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mathsf{A}\varphi,$$

where $p \in \mathscr{P}$. Other Boolean connectives are obtained from $\neg$ and $\wedge$ in the usual way; $\Box\varphi$ is $\neg\Diamond\neg\varphi$; and $\mathsf{E}\varphi$ is $\neg\mathsf{A}\neg\varphi$. The members of $\mathscr{F}$ are *formulas*. Lowercase Roman letters indicate proposition symbols, lowercase Greek letters indicate formulas, and uppercase Greek letters indicate sets of formulas. Let $\varphi \in \mathscr{F}$, the language of φ, notation $\mathscr{L}(\varphi)$, is its set of propositional symbols.

The semantics of KDA is defined in terms of *Kripke models* that are *serial*. A Kripke model $\mathfrak{M}$ is a tuple $\langle W, R, v\rangle$ where: W is a set of elements (or *worlds*); $R \subseteq W \times W$ is the *accessibility relation*; and $v : W \to \wp(\mathscr{P})$ is the *valuation function*. A Kripke model is *serial* if for every $w \in W$, there is $w' \in W$ s.t. wRw'. Henceforth, we assume that all Kripke models are serial.

Let $\mathfrak{M} = \langle W, R, v\rangle$ be a Kripke model, $w \in W$, and $\varphi \in \mathscr{F}$, define the *satisfiability relation* $\mathfrak{M}, w \models \varphi$ according to the rules below.

$$\begin{array}{lll} \mathfrak{M}, w \models p & \text{iff} & p \in v(w) \\ \mathfrak{M}, w \models \Diamond\varphi & \text{iff} & \text{there is } w' \in W \text{ s.t. } wRw' \text{ and } \mathfrak{M}, w' \models \varphi \\ \mathfrak{M}, w \models \mathsf{A}\varphi & \text{iff} & \text{for all } w' \in W,\ \mathfrak{M}, w' \models \varphi. \end{array}$$

We omit Boolean connectives. A Kripke model $\mathfrak{M} = \langle W, R, v\rangle$ *satisfies* a set of formulas Φ *at a world* $w \in W$, notation $\mathfrak{M}, w \models \Phi$, if $\mathfrak{M}, w \models \varphi$ for all $\varphi \in \Phi$. And it *validates* Φ, notation $\mathfrak{M} \models \Phi$, if $\mathfrak{M}, w \models \Phi$ for all $w \in W$.

Two reasonable notions of *modal logical consequence* between sets of formulas (i.e., premisses), and formulas (i.e., their consequences) are: *global* and *local modal logical consequence*, notation $\models^g$ and $\models^l$, respectively. More precisely, $\Phi \models^g \varphi$ if for every $\mathfrak{M}$, if $\mathfrak{M} \models \Phi$, then, $\mathfrak{M} \models \varphi$. And $\Phi \models^l \varphi$ if for every $\mathfrak{M}$ and w in $\mathfrak{M}$, if $\mathfrak{M}, w \models \Phi$, then, $\mathfrak{M}, w \models \varphi$. The global modality A enables us to handle global and local modal logical consequence uniformly, i.e., $\Gamma \models^g \varphi$ iff $\mathsf{A}(\Gamma) \models^l \varphi$, where $\mathsf{A}(\Gamma) = \{\mathsf{A}\gamma \mid \gamma \in \Gamma\}$ (see [11]). For this reason, we define semantic consequence as local modal logical consequence and drop the superscript l. We write $\models \varphi$ if $\emptyset \models \varphi$. If Γ is finite, $\Gamma \models \varphi$ iff $\models (\bigwedge\Gamma) \supset \varphi$.

We are particularly interested in *interpolation*. This property comes in many flavours [12]. Def. 2.1 introduces some commonly found formulations.

Definition 2.1 [Interpolation] The consequence relation $\models$ has the:

AIP *arrow interpolation property* if whenever $\Phi \models \varphi \supset \psi$, there exists θ s.t.: $\Phi \models \varphi \supset \theta$, $\Phi \models \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

TIP *turnstile interpolation property* if whenever $\Phi \cup \{\varphi\} \models \psi$, there is θ s.t.: $\Phi \cup \{\varphi\} \models \theta$, $\Phi \cup \{\theta\} \models \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

SIP *split interpolation property* if whenever $\Phi \cup \{\varphi_1, \varphi_2\} \models \psi$, there is θ s.t.: $\Phi \cup \{\varphi_1\} \models \theta$, $\Phi \cup \{\varphi_2, \theta\} \models \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi_1) \cap (\mathscr{L}(\varphi_2) \cup \mathscr{L}(\psi))$.

The formula θ in AIP, TIP, and SIP is an *interpolant*.

In FOL, AIP, TIP, and SIP are equivalent to each other. In general this may not be the case (depending on both compactness and the deduction theorem). With the local consequence relation, AIP, TIP and SIP are all equivalent. With the global consequence relation this equivalence might not hold. The moral of the story: attention must be paid to the precise formulation of interpolation.

3 Default Logic over KDA

The set $\mathscr{D}$ of *default rules* over $\mathscr{F}$ contains all figures $\pi : \rho \,/\, \chi$ for $\{\pi, \rho, \chi\} \subseteq \mathscr{F}$. The members of $\mathscr{D}$ are *default rules*. The formula π is called *prerequisite* of the default rule, ρ its *justification*, and χ its *consequent*. We single out Δ for sets of default rules and δ for default rules. For $\Delta \subseteq \mathscr{D}$, $\Pi(\Delta) = \{\pi \mid \pi : \rho \,/\, \chi \in \Delta\}$, $P(\Delta)$ and $X(\Delta)$ are defined similarly for justifications and consequents, resp.

The set $\mathscr{P}$ contains all tuples $\langle \Phi, \Delta, \prec \rangle$, where $\Phi \subseteq \mathscr{F}$, $\Delta \subseteq \mathscr{D}$, and $\prec$ is a partial order on Δ. The members of $\mathscr{P}$ are *(default) premiss sets*. We restrict our attention to cases in which Φ and Δ are finite. $\mathscr{P}$ enables a presentation of a consequence relation for default reasoning over KDA and a justification of such a consequence relation in terms of *extensions*. In this respect, there are two options. For $\varphi \in \mathscr{F}$, φ is a *sceptical default consequence* of $\langle \Phi, \Delta, \prec \rangle \in \mathscr{P}$, notation $\langle \Phi, \Delta, \prec \rangle \mathrel{|\!\!\approx^s} \varphi$, if for *every* extension E of $\langle \Phi, \Delta, \prec \rangle$, $E \models \varphi$. Or φ is a *credulous default consequence* of $\langle \Phi, \Delta, \prec \rangle \in \mathscr{P}$, notation $\langle \Phi, \Delta, \prec \rangle \mathrel{|\!\!\approx^c} \varphi$, if for *some* extension E of $\langle \Phi, \Delta, \prec \rangle$, $E \models \varphi$. In any case, an extension may be seen as an interpretation structure of a syntactical kind (i.e., an extension is a premiss set in KDA taking the usual role of a model).

We skip the formal definition of an extension for the sake of brevity and list some of its properties. An extension of $\langle \Phi, \Delta, \prec \rangle$ is a finite subset of formulas including Φ and closed under the application of default rules. The criterion of application of a default rule is that of Łukaszewicz [13]. Default rules are selected for application in the order in which they appear in a linear extension of $\prec$. This defines a priority relation on default rules [7]. This priority relation differs from some standard approaches in that default rules with lower priority are not included in an extension if this depends on default rules with higher priority. A default premiss set always has one extension but it might have more then one. The set of all extensions of a premiss set is indicated by $\mathscr{E}(\Phi, \Delta, \prec)$.

Let $\models\!\approx$ be either $\models\!\approx^s$ or $\models\!\approx^c$, monotonicity for $\models\!\approx$ is: if $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi$, then $\langle \Phi \cup \Phi', \Delta \cup \Delta', (\prec \cup \prec')^* \rangle \models\!\approx \varphi$. The relation $\models\!\approx$ is non-monotonic.

4 Interpolation in $\mathscr{D}$KDA

It is well known that (local) consequence for KDA has AIP (and hence, TIP and SIP). We now discuss how this affects the interpolation property of the non-monotonic consequence relation $\models\!\approx$.

Definition 4.1 The default consequence relation $\models\!\approx$ has the: *arrow interpolation property* (AIP) if whenever $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \psi$, there is θ s.t.: $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \theta$, $\langle \Phi, \Delta, \prec \rangle \models\!\approx \theta \supset \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$. θ is an *interpolant.*

Proposition 4.2 $\models\!\approx^c$ *has the AIP.*

It is not straightforward to prove whether the AIP holds for $\models\!\approx^s$; and if not, whether a weaker form of this property holds. This said, $\models\!\approx^s$ has the following easily established "interpolation" property.

Definition 4.3 The default consequence relation $\models\!\approx$ has the: $\vee\wedge$-*interpolation property* (OAIP) if whenever $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \psi$, there are θ and θ' s.t.: $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi \supset \theta$, $\langle \Phi, \Delta, \prec \rangle \models\!\approx \theta' \supset \psi$, and $\mathscr{L}(\{\theta, \theta'\}) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

Proposition 4.4 $\models\!\approx^s$ *has the OAIP.*

Obviously, if $\models\!\approx$ has the AIP, it has the OAIP. What is interesting about Prop. 4.4 is that θ can be taken to be $\bigvee \theta_i$ and θ' to be $\bigwedge \theta_i$, where each θ_i is an interpolant at the level of extensions of the premiss set.

It is not difficult to formulate versions of the turnstile and the split interpolation properties for $\models\!\approx$; see below.

Definition 4.5 The consequence relation $\models\!\approx$ has the:

TIP *turnstile interpolation property* if whenever $\langle \Phi \cup \{\varphi\}, \Delta \rangle \models\!\approx \psi$, there is θ s.t.: $\langle \Phi \cup \{\varphi\}, \Delta \rangle \models\!\approx \theta$, $\langle \Phi \cup \{\theta\}, \Delta \rangle \models\!\approx \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi) \cap \mathscr{L}(\psi)$.

SIP *split interpolation property* if whenever $\langle \Phi \cup \{\varphi_1, \varphi_2\}, \Delta \rangle \models\!\approx \psi$, there exists θ s.t.: $\langle \Phi \cup \{\varphi_1\}, \Delta \rangle \models\!\approx \theta$, $\langle \Phi \cup \{\varphi_2, \theta\}, \Delta \rangle \models\!\approx \psi$, and $\mathscr{L}(\theta) \subseteq \mathscr{L}(\varphi_1) \cap (\mathscr{L}(\varphi_2) \cup \mathscr{L}(\psi))$.

There is an interesting challenge to Def. 4.5: *cumulativity.* This property states: if $\langle \Phi, \Delta, \prec \rangle \models\!\approx \varphi$ and $\langle \Phi, \Delta, \prec \rangle \models\!\approx \psi$, then, $\langle \Phi \cup \{\varphi\}, \Delta, \prec \rangle \models\!\approx \psi$. Cumulativity fails for $\models\!\approx$. Since TIP and SIP accumulate the interpolant as a sentence in the premiss set, cumulativity might hinder interpolation results in these cases.

5 Discussion and Final Remarks

We discussed interpolation properties for $\mathscr{D}\mathsf{KDA}$. Because of the non-monotonic nature of $\mathscr{D}\mathsf{KDA}$ and the composite structure of its premiss sets, some standard ideas are not applicable in this framework. Our preliminary results are mainly concerned with the definition of adequate notions of interpolation for this logic. As a first step, we took advantage of interpolation results of the underlying modal logic to obtain new notions of interpolation for $\mathscr{D}\mathsf{KDA}$. There are many open questions, such as studying possible relations between the interpolants θ and θ' in Def. 4.3. Another interesting question concerns the definition of interpolation properties for $\models\!\approx^s$ which do not look into the internal structure of the extensions of a premiss set. It would also be interesting to study the relation between interpolation and the property of cumulativity.

In this paper we focused on $\mathscr{D}\mathsf{KDA}$, but the interpolation notions introduced for this particular case would hopefully be relevant for other default versions of modal logics.

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Verified synthesis of (very simple) Sahlqvist correspondents via Coq

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Abstract

We provide an account of our formalisation of Sahlqvist's global correspondence theorem for the very simple Sahlqvist class in the proof-assistant Coq. We constructed our own encodings of modal, first-order and second-order fragments and provide corresponding libraries containing numerous lemmata required for the proof of (very simple) Sahlqvist's theorem. Moreover, we extracted from the constructive Coq proof code a verified program in Haskell that computes the first-order correspondent given a very simple Sahlqvist modal formula. We believe this verified program is the first of its kind in this area and we hope that this first case study will pave the way for future formalisation work to be done in correspondence theory and beyond.

Keywords: Sahlqvist's theorem, interactive theorem proving, proof-assistant, Coq, verified program, extraction, synthesis.

1 Introduction

Sahlqvist's correspondence theorem [10], proved in 1975, is a fundamental result in modern modal logic which states that from every modal formula φ from the syntactically defined class of "Sahlqvist formulae", we can compute a first-order formula that describes exactly the class of Kripke frames on which φ is valid for all valuations. Since then, various algorithms and computer implementations (for example DLS [6], SCAN [7], SQEMA [3], ALBA [4]) have been developed to compute first-order correspondents using techniques ranging from second-order quantifier elimination, which is at the heart of Sahlqvist's proof, through to algebraic tools. Are these pen-and-paper proofs and algorithms correct? Even with correct proofs, how do we know that the implementations are correct?

We have used the proof-assistant Coq to formalise a version of Sahlqvist's global correspondence theorem to guarantee correctness of the proofs. We have followed the breakdown in Blackburn, de Rijke and Venema [2], having successfully finished the very simple Sahlqvist case and intending to finish the full Sahlqvist case. We have created, from scratch, Coq libraries for reasoning

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about second-order, first-order and modal logic. Finally, we have successfully used Coq's extraction facilities to produce a verified Haskell program that computes correspondents of very simple Sahlqvist formulae - the first of its kind. Our Coq code is here: `https://github.com/caitlindabrera/Sahlqvist/`

2 Introducing Coq

Proof-assistants are computer programs that allow users to encode the relevant mathematical definitions and the theorem to be proved, and prove the theorem by directing the proof-assistant using previously established theorems and proof rules. Additionally, some can synthesise computer programs from constructive proofs. We used the proof-assistant Coq (version 8.5pl3; October, 2016) which has this extraction facility. But why should we trust it?

The mathematics underpinning Coq versions 8.0 and above is the Predicative Calculus of (Co)Inductive Constructions (PCIC), which is a higher-order typed lambda calculus that extends the Calculus of Constructions with inductive types. A brief introduction [9] and a thorough treatment [1] are available.

Trust rests on three main aspects: the OCaml Coq-kernel implementation; the OCaml compiler; and our computer hardware [5]. The kernel OCaml implementation has been extensively used and scrutinised by numerous experts so it is unlikely to contain major bugs: minor bugs have been reported and repaired almost immediately. Moreover, Coq produces a proof-script that can be checked by other proof-assistants if required. Bugs in the OCaml compiler are beyond our control, but we can always use multiple different compilers. Lastly, Coq and our proofs can simply be run on multiple computers that use different hardware. Our encodings must correctly express the underlying definitions and theorem, but readers need not check (or understand) the Coq proof.

Our task in formalising Sahlqvist's theorem then is to encode the syntax and semantics of modal logic, the syntax and semantics of fragments of first- and second-order logic (particulars of the fragments described shortly), as well as the statement of Sahlqvist's theorem using these mechanisms. From now on, we elide "fragment" and use "first-order" and "second-order" logic.

3 Theorem statement

We state the theorem in Coq and explain its constituents.

```
Theorem vsSahlq_full_Modal : forall phi, vsSahlq phi ->
existsT (alpha : SecOrder), is_unary_predless alpha = true
/\ forall W Iv Ip Ir,
        SOturnst W Iv Ip Ir alpha <-> mturnst_frame W Ir phi.
```

Our encoding is very similar to the pen-and-paper expression: for every modal formula, `phi`, if `phi` has the syntactic form of a very simple Sahlqvist formula (captured by `vsSahlq`), then there exists a second-order formula, `alpha`, which is a first-order frame condition (captured by `is_unary_predless`), and `phi` and `alpha` are equivalent in a sense captured by their respective semantics (using the `SOturnst` and `mturnst_frame` functions).

We elide many of the definitions here for space reasons, but make several notes. Modal formulae are defined in the usual way using propositional variables and connectives `mneg`, `mconj`, `mdisj`, `mimpl`, `box` and `diam` for $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\Box$ and $\Diamond$, respectively. The function `vsSahlq` requires that `phi` is a very simple Sahlqvist formula, the syntactically defined class of modal formulae for which we have proven the theorem statement. The type `SecOrder` captures the notion of the usual definitions for our fragment of second-order formulae. The function `is_unary_predless` specifies that `alpha` contains no unary predicates, making it a first-order frame condition. The statement `mturnst_frame W Ir phi` encodes the forcing relation $\langle W, R\rangle \Vdash \varphi$ between frames and modal formulae, with the usual hidden quantification over valuations and worlds: why we use `Ir` for R is explained below. Finally, the function `SOturnst` encodes the satisfaction relation $W, I_v, I_p, I_r \vDash \alpha$ on second-order structures, taking in a set W, and three interpretation functions I_v, I_p and I_r on first-order variables, unary predicates and the single binary predicate, respectively. Thus `Ir` can be considered as both a binary relation and an interpretation function over the single binary predicate of our second-order fragment.

The reason why we specify that `phi` is second-order with the further property of being first-order (via `is_unary_predless`) is because the proof goes via second-order logic. It starts with a modal formula, converts that to an equivalent distinctly second-order formula via the standard translation and then works on that second-order formula to produce an equivalent first-order formula. The modal to second-order equivalence part of the proof is easy: it is proven by the correctness of the standard translation. From second-order to first-order involves just instantiating the present universal quantifiers which is obvious. The reverse direction, from first-order to second-order is the interesting bit mathematically, essentially proving a theorem of the form $A(P_1, ..., P_n) \rightarrow \forall P_1 ... \forall P_n . A(P_1, ..., P_n)$, where $A(P_1, ..., P_n)$ is a first-order formula with the only free unary predicates being $P_1, ..., P_n$.

While this is the only "difficult" part in the pen-and-paper proof, the Coq proof requires a lot of second-order manipulations that result in equivalent formulae. A small example is the need to pull out existential quantifiers in an antecedent, transforming $((\exists x.\alpha) \rightarrow \beta)$ to $(\forall x.(\alpha \rightarrow \beta))$, for suitable x, α and β. Despite barely getting a mention in most pen-and-paper proofs of the theorem, many complicated examples arose from nested renaming functions regarding first-order variables, and these constituted a significant portion of the work. One reason was the lack of any previous libraries for such manipulations.

Thus we have produced two libraries, one for modal logic and the other for second-order logic. Note though that the latter library only encodes notions for the fragment of second-order logic needed for the proof: monadic second-order logic with equality over first-order variables plus a single binary relation (that corresponds to the binary relation of the model frames).

4 Synthesising first-order correspondents via extraction

If the Coq proof of a theorem uses purely constructive reasoning (as opposed to classical) then it is amenable to Coq's code extraction feature. Our formalised proof captured computational content in this way and so we extracted the Haskell code that computes the first-order correspondents given a very simple Sahlqvist formula. This extraction produced 21 Haskell modules, to which we added a 22nd called Main, concerned purely with printing and readability.

In the following examples, the modal implication is `-->`, diamond is `dia` and `corr` is the function that computes the correspondents.

Example 4.1 The first-order correspondent of $p \to \Diamond p$ is $\forall x.R(x,x)$, which captures reflexivity. The call to `corr ((p 1) --> (dia (p 1)))` produces

```
(all x0.((x0 = x0)-->(ex x2.(R(x0, x2) /\ (x2 = x0))))).
```

Example 4.2 The first-order correspondent of $\Diamond\Diamond p \to \Diamond p$ is $\forall x.\forall y.\forall z.R(x,y) \wedge R(y,z) \to R(x,z)$, which captures transitivity. The call to `((dia (dia (p 1))) --> (dia (p 1)))` produces:

```
(all x0.(all x4.(all x5.(((R(x0, x4) /\ R(x4, x5)) /\ (x5 = x5))
    -->  (ex x6.(R(x0, x6) /\ (x6 = x5))))))).
```

Our Haskell program outputs an unnecessarily complicated formula with existential quantifiers but they are equivalent to the usual correspondents that we know. Our tests on a small number of a variety of formulae have all taken less than one second to compute, thus we do not envisage problems with efficiency.

5 Further work

We have proved very simple Sahlqvist's theorem in Coq and produced a verified program that computes the first-order correspondent. We have almost completed the simple Sahlqvist case with intentions to continue on to the full case. We also want to prove the stronger local version of Sahlqvist's theorem which implies the global version we have proved in our formalisation. The difficulties surrounding this task again relate to first-order variable names and nested renaming functions, rather than mathematically interesting properties.

As listed in the introduction, there are numerous classes of formulae with computable first-order correspondents and various algorithms, all of which have potential to be formalised with the immense benefit of verified programs. Kracht's theorem [8], which is a reverse of Sahlqvist's theorem that takes in a specific kind of first-order formula and produces a modal correspondent.

Other avenues for further work are: improving our current code, including a function to simplify the correspondent according to some kind of measure; reconsidering the encodings of first-order variables to minimise the amount of second-order massaging required; and refining and extending our second-order library from the fragment we used to the full second-order logic to be able to handle other logic contexts and to allow others to follow in our footsteps.

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Provability logic meets the knower paradox

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1 Introduction

The knower paradox revolves around the following statement:

$$\text{We know that statement (P) is false.} \tag{P}$$

The paradoxical reasoning goes as follows: Suppose (P) is true. Since it states that 'we know statement (P) is false', we know statement (P) is false. Since knowledge implies truth (a usual epistemological assumption), it follows that (P) is false, thus leading to a contradiction. Hence (P) is false and since *we* were the ones who just proved it, we know it to be false. However, this is exactly what (P) states, therefore it is true. So we have arrived at a paradox.

The knower paradox was first formulated by Kaplan and Montague [6]. They used *elementary syntax*, by which they understood "a first-order theory containing (...) all standard names (of expressions), means for expressing syntactical relations between, and operations on, expressions, and appropriate axioms involving these notions" [6, Footnote 10, p. 89]. Note that by elementary syntax they meant both a formal language and some sort of proof system. Kaplan and Montague used '$\varphi \vdash \psi$' to express that ψ is derivable from φ within the theory and '$\vdash \varphi$' means that φ is provable within this theory. In addition, they used names for expressions, where $\overline{\varphi}$ denotes the name of expression φ. The following two formulae are added to the elementary syntax:

$K(\overline{\varphi})$	A knows the expression φ
$I(\overline{\varphi}, \overline{\psi})$	$\varphi \vdash \psi$

According to Kaplan and Montague [6, p. 87], we can now formalize (P):

$$\vdash D \leftrightarrow K(\overline{\neg D}).$$

From this expression, some version of the knower paradox is derived, if the following three assumptions are made:

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$$E1 := K(\overline{\neg D}) \to \neg D \tag{E1}$$
$$E2 := K(\overline{E_1}) \tag{E2}$$
$$E3 := [I(\overline{E_1}, \overline{\neg D}) \land K(\overline{E_1})] \to K(\overline{\neg D}) \tag{E3}$$

We derive the knower paradox as follows:

(1)	$\vdash D \leftrightarrow K(\overline{\neg D})$	by definition of D
(2)	$\vdash D \to K(\overline{\neg D})$	by (1), PC
(3)	$E1 \vdash K(\overline{\neg D}) \to \neg D$	by definition of $E1$
(4)	$E1 \vdash D \to \neg D$	by (2), (3), HS
(5)	$E1 \vdash \neg D$	by (4), PC
(6)	$E1 \vdash I(\overline{E_1}, \overline{\neg D})$	(5), by definition of I
(7)	$E1, E2 \vdash K(\overline{E1})$	by definition of $E2$
(8)	$E1, E2 \vdash I(\overline{E_1}, \overline{\neg D}) \land K(\overline{E_1})$	by (6), (7), PC
(9)	$E1, E2, E3 \vdash [I(\overline{E_1}, \overline{\neg D}) \land K(\overline{E_1})] \to K(\overline{\neg D})$	by definition of $E3$
(10)	$E1, E2, E3 \vdash K(\overline{\neg D})$	by (8), (9), MP
(11)	$E1, E2, E3 \vdash K(\overline{\neg D}) \to D$	by (1), PC
(12)	$E1, E2, E3 \vdash D$	by (10), (11), MP
(13)	$E1, E2, E3 \vdash \neg D \to D$	by (12), PC
(14)	$E1, E2, E3 \vdash D \leftrightarrow \neg D$	by (4), (13), PC

Over the years, many solutions have been proposed. We take Paul Égré's paper [4] as our point of departure. He argues that the knower paradox is solvable when modal provability logic is applied and uses three different interpretations of provability logic to solve the paradox, of which we focus on the one inspired by Solovay [9]. Provability logic provides a natural system in which both modalities and self-reference are treated rigorously, hence it may shed some light on where the paradoxical reasoning underlying the paradox goes awry. Our main contribution is an assessment of how the interpretation by Solovay fares in the light of Susan Haack's criteria for solutions to paradoxes [5], which include both technical and philosophical desiderata. In this way we hope to advance the debate regarding the knower paradox.

2 Provability Logic and Formal Systems of Arithmetic

We give a reminder of arithmetic and the provability logics **GL** and **GLS**.

2.1 Robinson Arithmetic and Peano Arithmetic

Here follow the axioms of Robinson arithmetic Q [8]: $\forall x(0 \neq Sx); \forall x \forall y(Sx = Sy \to x = y); \forall x(x \neq 0 \to \exists y(x = Sy)); \forall x(x + 0 = x); \forall x \forall y(x + Sy = S(x + y)); \forall x(x \cdot 0 = 0); \forall x \forall y(x \cdot Sy = (x \cdot y) + x)$. A statement φ is a theorem of Q if it is (an instance of) an axiom or can be derived from the axioms by the available rules of inference, modus ponens and generalization. Peano arithmetic (PA) [7] extends Q by the *Induction Schema*:$\{\varphi(0) \land \forall x(\varphi(x) \to \varphi(Sx))\} \to \forall x \varphi(x)$.

Theorem 2.1 (Diagonal Lemma, [2, p. 54] [3]) *Suppose that $P(y)$ is a formula of the language of* PA *in which no variable other than y is free. Then there exists a sentence S of the language of* PA *such that* $\mathrm{PA} \vdash S \leftrightarrow P(\overline{S})$.

2.2 Provability Logic

The provability logic **GL** contains the following axioms:

$$\text{All (instances of) propositional tautologies} \tag{A1}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{A2}$$

$$\Box(\Box\varphi \to \varphi) \to \Box\varphi \tag{GL}$$

The rules of inference of **GL** are modus ponens and necessitation (if $\varphi \in \mathbf{GL}$, then $\Box\varphi \in \mathbf{GL}$). Note that $\Box\varphi \to \Box\Box\varphi \in \mathbf{GL}$ [10].

2.3 The Relation between Provability Logic and Peano Arithmetic

A *realization* is a function $*$ assigning to each propositional atom of modal logic a sentence of the language of arithmetic, inductively defined by: $\bot^* = \bot; (\varphi \to \psi)^* = (\varphi^* \to \psi^*); (\Box\varphi)^* = Prov(\overline{\varphi^*})$. Solovay [9] proved that **GL** is arithmetically complete with respect to PA. The arithmetical soundness of **GL** was already clear. Formally:

$$\mathbf{GL} \vdash \varphi \text{ if and only if PA } \vdash \varphi^* \text{ for all realizations } *.$$

The system **GLS**, defined by Solovay [9, Section 5.1] [4], contains all theorems of **GL** as axioms as well as all instances of the reflection principle $\Box\varphi \to \varphi$, and modus ponens is its single rule of inference. The system **GLS** enjoys arithmetical soundness and completenesswith respect to the standard model:

$$\mathbf{GLS} \vdash \varphi \text{ if and only if } \langle \omega; +, \cdot \rangle \models \varphi^* \text{ for all realizations } *.$$

Égré [4, p. 43] defines PA^+ as the closure under *modus ponens* of PA, supplemented with all instances of the reflection principle $Prov(\overline{A}) \to A$. PA^+ is stronger than PA because it can now prove the consistency of PA as an instance of reflection: $Prov(\overline{\bot}) \to \bot$. It is not hard to prove that we also have: [5]

$$\mathbf{GLS} \vdash \varphi \text{ if and only if } \mathrm{PA}^+ \vdash \varphi^* \text{ for all realizations } *.$$

3 Solving the knower paradox using provability logic

We consider Solovay's **GLS**, which solves the knower paradox according to Égré [4]. We then discuss whether the solution satisfies Haack's criteria [5].

Why is the knower paradox prevented in **GLS**? Remember that $K(\overline{E1})$, where $E1$ was defined as $K(\overline{\neg D}) \to \neg D$, was needed in Kaplan and Montague's derivation of the knower paradox [6] (p. 32, Step (7)). In **GLS**, we have $\Box\neg D \to \neg D$ as an instance of the reflection principle. Because necessitation is not an inference rule of **GLS**, $\Box(\Box\neg D \to \neg D)$ cannot be derived, therefore, Kaplan and Montague's derivation cannot be repeated in **GLS**. Let us now assess to which extent Solovay's **GLS** satisfies Haack's criteria for solutions to paradoxes.

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[4] We follow current conventions [2] in that Solovay's G is our **GL** and his G' is our **GLS**.

[5] The proof is in our journal manuscript under revision, "Solutions to the knower paradox in the light of Haack's criteria".

3.1 The Formal Part of Solovay's Theory as a Solution

Haack's first requirement on solutions to paradoxes says that a solution should contain a consistent formal system indicating an unacceptable premise, principle of inference, or set of theorems [5]. Solovay's formal system **GLS** indicates the rejection of $K(\overline{K(\overline{\varphi}) \to \varphi})$, which is achieved by disallowing the necessitation rule to be applied to the reflection principle $K(\overline{\varphi}) \to \varphi$. Is **GLS** consistent? Solovay [9] proved that **GLS** is arithmetically sound with respect to the standard model. Since truth in a model implies consistency, **GLS** is consistent. So Haack's first requirement is satisfied.

3.2 The Philosophical Part of Solovay's Theory as a Solution

Haack's second requirement says that solutions to paradoxes should provide a philosophical explanation why the suspect premise or principle of inference seems acceptable but is not. So in the case of **GLS**, there needs to be an argument for rejecting $K(\overline{K(\overline{\varphi}) \to \varphi})$ and/or for disallowing the necessitation rule to apply to the reflection principle $K(\overline{\varphi}) \to \varphi$. This argument should be independent of the existence of the knower paradox. Solovay [9] did not consider **GLS** in the context of the knower paradox. His article is about provability, not knowledge, so it does not contain arguments for rejecting $K(\overline{K(\overline{\varphi}) \to \varphi})$ itself.

There are indeed independent reasons to reject $Prov(\overline{Prov(\overline{\varphi}) \to \varphi})$. The formalized version of Löb's theorem states that PA $\vdash Prov(\overline{Prov(\overline{\varphi}) \to \varphi}) \to Prov(\overline{\varphi})$. So if $Prov(\overline{Prov(\overline{\varphi}) \to \varphi})$ were accepted as an axiom scheme, then $Prov(\overline{\varphi})$ would hold for every statement φ, even for false statements.

Égré [4, p. 42] argues that **GL** can be seen as a "system formalizing the knowledge of an ideal mathematician recursively generating all the theorems of PA and reflecting on the scope of his knowledge". However, the only reason mentioned in [4] for accepting **GLS** is not independent of the existence of the paradox: the necessitation rule is not allowed to be applied to the reflection principle ($K(\overline{\varphi}) \to \varphi$) just to prevent the paradox. Therefore, for the solution to satisfy Haack's second requirement, we will need reasons to let a knowledge predicate satisfy the axioms of **GLS** independently of the knower paradox.

Let us therefore attempt to give an independent reason. Since **GLS** is arithmetically sound with respect to the standard model $\langle \omega; +, \cdot \rangle$, for every formula φ in the language of **GL**, **GLS** $\vdash \Box\varphi$ implies $\omega \models Prov_{\mathrm{PA}}(\overline{\varphi^*})$, so there exists a proof of φ^* in PA, therefore PA $\vdash \varphi^*$ for every realization $*$. So for $\Box$ interpreted as knowledge, **GLS** is *epistemically conservative* over PA, meaning that **GLS** will not prove any 'new' formulas of the form 'It is known that φ', i.e. $\Box\varphi$, for which PA does not prove φ^* yet (cf. [3]). This is an argument to accept the theory **GLS** as a solution to the knower paradox.

3.3 The Scope of Solovay's Theory as a Solution

Haack's third requirement states that a solution to a paradox should not be too broad or too narrow. Solovay's system **GLS** is consistent, so it does not prove just everything. On the other hand, it does prove other desired theorems than just those needed to formalize the knower paradox. For example, Gödel sentence G in the language of PA satisfies PA $\vdash G \leftrightarrow \neg Prov(\overline{G})$. Is there a sentence G satisfying **GLS**$\vdash G \leftrightarrow \neg\Box G$? Yes there is, namely $\neg\Box\bot$. The formula $\neg\Box\bot \leftrightarrow \neg\Box\neg\Box\bot$ is provable in **GL**, and thus in **GLS** (see [10, Section 2.2]). Therefore, **GLS** is neither too broad nor too narrow.

3.4 Discussion: Interpreting Knowledge as Provability

Why may it seem at first sight plausible to interpret knowledge (facts about Peano arithmetic known by some mathematicians) as provability in PA? According to many non-Platonists, proofs are constructed by mathematicians: there exists a proof of a certain statement only if there has been a mathematician who proved it and therefore knows it.

However, Platonists can argue against interpreting knowledge as provability that knowledge depends on mathematicians and on time, while provability does not. There are statements known by mathematicians but not provable in PA, such as the Gödel sentence for PA. Conversely, a theorem provable in PA but for a long time not known is the formalized version of Löb's theorem.

Justification logic provides an interesting formal take on the correspondence between knowledge and provability. It includes *explicit knowledge* $t : \varphi$, meaning that 'φ is justified by t'. A logic combining implicit and explicit knowledge is **S4LP** [1], containing the axiom scheme $(t : \varphi) \to \Box\varphi$. Since a proof is a form of justification, this logic is a way to connect knowledge and provability.

3.5 Conclusion

Summarizing the discussion about the quality of Solovay's system **GLS** as a solution to the knower paradox, Haack's first requirement is clearly satisfied and the solution (after our addition to [4]) satisfies the second criterion. The third criterion is provisionally met, because the solution is not too narrow and provisionally not too broad.

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Sorting out FDE-based modal logics

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There is by now a vast literature on many-valued modal logics, among which a special place occupy four-valued modal logics. In turn, a lot of four-valued modal logics are built around first-degree entailment FDE and Belnap-Dunn logic [2,7]. The current study was initially aimed at solving a couple of specific problems: namely, providing simple alternative axiomatization of Lou Goble's modal logic $\mathsf{KN4}$ [8] as well as clarifying the relationship between systems BK and $\mathsf{BK}^{\mathsf{FS}}$ introduced in [12] and [13], respectively. In the process it became apparent that a lot of FDE-based modal logics are really not that different from the standpoint of modal operators themselves, a point rather obscured by different non-modal languages, semantics and axiomatic systems involved in working with them. To make this more transparent we want to build a theory of FDE-based modal logics from scratch in a way that is more general than that discussed by Priest in [14], where he introduced his system $\mathsf{K}_{\mathsf{FDE}}$. The main goal is to provide a uniform way of axiomatizing FDE-based modal logics which would allow to separate the behavior of modalities from other connectives as much as possible. The approach is very much inspired by those of [6] and [3,5]. During this first stage of the work we restrict ourselves to systems which have a classical flavor, that is we only show how to add classical conditionals: either weak implication $\to$ or strong implication $\Rightarrow$. This bypasses a lot of modal system based on the variants of Nelson's logic with constructible falsity [9]. Naturally, extending these results by adding intuitionistic implication is then an open research problem. We also (mostly) restrict ourselves to working with positive modal operators of necessity and possibility, leaving their negative counterparts for future research. Aside from the systems mentioned above we will also show how one variant of modal bilattice logic [15] can be integrated into our framework.

Throughout the paper we will assume that any language contains connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (negation). Following Priest [14]

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we start with the system FDE of first-degree entailment [7].

$a1_{fde}$. $\varphi \wedge \psi \vdash \varphi$;	$a2_{fde}$. $\varphi \wedge \psi \vdash \psi$;
$a3_{fde}$. $\varphi \vdash \varphi \vee \psi$;	$a4_{fde}$. $\psi \vdash \varphi \vee \psi$;
$a5_{fde}$. $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$;	$a6_{fde}$. $\varphi \dashv\vdash \sim\sim \varphi$;
$a7_{fde}$. $\sim (\varphi \wedge \psi) \dashv\vdash \sim \varphi \vee \sim \psi$;	$a8_{fde}$. $\sim (\varphi \vee \psi) \dashv\vdash \sim \varphi \wedge \sim \psi$;
$r1_{fde}$. $\varphi \vdash \psi; \psi \vdash \chi / \varphi \vdash \chi$;	$r2_{fde}$. $\varphi \vdash \chi; \psi \vdash \chi / (\varphi \vee \psi) \vdash \chi$;
$r3_{fde}$. $\chi \vdash \varphi; \chi \vdash \psi / \chi \vdash (\varphi \wedge \psi)$;	

where $\varphi \dashv\vdash \psi$ denotes a pair of sequents $\varphi \vdash \psi$ and $\psi \vdash \varphi$. It is well known that FDE does not have any theorems, which means that one cannot reduce it to its set of theorems (as in Hilbert-style systems). On the other hand, it is not quite a Gentzen-style sequent system. It is rather somewhere in-between: an axiom system which aims to axiomatize some elementary (single-premiss single-conclusion) inferences, while minimizing the set of rules. Thus by a logic L we will understand a set of sequences of the form $\varphi \vdash \psi$, which contains all axioms and is closed under all inference rules of FDE. We denote by $\varphi \vdash_L \psi$ the fact that $\varphi \vdash \psi$ belongs to L. We say that a formula φ is *(dual) theorem* of L if $\chi \vdash_L \varphi$ ($\chi \vdash_L \sim \varphi$) holds for every formula χ and we say that φ is a *(dual) counter-theorem* if $\varphi \vdash_L \chi$ ($\sim \varphi \vdash_L \chi$) holds for every χ. We use t, dt, ct and dct to denote some generic theorem, dual theorem, counter-theorem or dual counter-theorem, respectively, assuming it exists.

It is quite easy to extend FDE with constants T (true), F (false), N (neither) and B (both) as well as weak and strong (classical) implication connectives $\to$ and $\Rightarrow$ and bilattice operators $\otimes$ and $\oplus$. For instance, $\to$ and $\otimes$ can be characterized by adding to FDE the following

$a1_{\to}$. $\varphi \wedge (\varphi \to \psi) \vdash \psi$;	$a2_{\to}$. $\chi \vdash \varphi \vee (\varphi \to \psi)$;
$a3_{\to}$. $\sim (\varphi \to \psi) \dashv\vdash (\varphi \wedge \sim \psi)$;	$r1_{\to}$. $\varphi \wedge \psi \vdash \chi / \varphi \vdash \psi \to \chi$;
$a1_{\otimes}$. $\varphi \otimes \psi \dashv\vdash \varphi \wedge \psi$;	$a2_{\otimes}$. $\sim (\varphi \otimes \psi) \dashv\vdash \sim \varphi \otimes \sim \psi$.

For some subset X of these connectives we denote by $\mathsf{FDE}(X)$ a logic obtained by extending FDE with the corresponding connectives. Naturally, there are some known logics among these systems, for instance $\mathsf{FDE}(\otimes, \oplus, \to)$ is the logic of bilattices (without constants) [1] and $\mathsf{FDE}(\Rightarrow)$ is Brady's $\mathsf{BD4}$ [4]. We fix some such X for the remainder of the paper.

One consequence of having strong negation is that adding a unary operator $\circ$ essentially results in adding two different operators — in general there is no relation between the behavior of $\circ$ when standing under influence of an even number of negations and $\circ$ when standing under influence of an odd number of negations. This point is made explicit in [11] where some operators which are only partially defined are introduced. To keep things as general as possible we first consider how to add to $\mathsf{FDE}(X)$ these partially defined operators as well as some natural fully defined ones.

- An operator $\Box$ is called a *v-box* in L if it satisfies: i) $\Box\varphi \wedge \Box\psi \vdash_L \Box(\varphi \wedge \psi)$; ii) $\varphi \vdash_L \Box t$; iii) $\varphi \vdash_L \psi / \Box\varphi \vdash_L \Box\psi$.

- An operator $\Box$ is called an *f-box* in L if it satisfies: i) $\sim\Box(\varphi\wedge\psi)\vdash_L\sim\Box\varphi\vee\sim\Box\psi$; ii) $\sim\Box cdt\vdash_L\varphi$; iii) $\sim\varphi\vdash_L\sim\psi/\sim\Box\varphi\vdash_L\sim\Box\psi$.
- An operator $\Diamond$ is called a *v-diamond* in L if it satisfies: i) $\Diamond(\varphi\vee\psi)\vdash_L\Diamond(\varphi\vee\psi)$; ii) $\Diamond ct\vdash_L\varphi$; iii) $\varphi\vdash_L\psi/\Diamond\varphi\vdash_L\Diamond\psi$.
- An operator $\Diamond$ is called a *f-diamond* in L if it satisfies: i) $\sim(\Diamond\varphi\vee\Diamond\psi)\vdash_L\sim\Diamond(\varphi\wedge\psi)$; ii) $\varphi\vdash_L\sim\Diamond dt$; iii) $\sim\varphi\vdash_L\sim\psi$ implies $\sim\Diamond\varphi\vdash_L\sim\Diamond\psi$.
- We say that $\Box$ is a *vf-box* if it is both a v- and an f-box and we say that $\Diamond$ is a *vf-diamond* if it is both a v- and an f-diamond.

These definitions immediately give us the way of extending $\mathsf{FDE}(X)$ by adding one of them. We denote by $\mathsf{FDE}(X;\Box^v)$, $\mathsf{FDE}(X;\Box^f)$, $\mathsf{FDE}(X;\Box^{vf})$ the result of extending $\mathsf{FDE}(X)$ with a v-box, f-box or vf-box, respectively, and similarly for $\Diamond$.

We turn to semantics. An *$\mathsf{FDE}(X,\circ^\alpha)$-frame* is a non-empty set together with an accessibility relation $R^+_\circ$ if $\circ$ is a v-modality and an accessibility $R^-_\circ$ if it is an f-modality (both if it is a vf-modality). An *$\mathsf{FDE}(X;\Box^+)$-model* is an $\mathsf{FDE}(X;\Box^+)$-frame together with two valuations v^+ and v^- which map all propositional variables and all formulas of the form $\circ\varphi$ to subsets of W. We omit validity conditions for non-modal connectives, validity conditions for modalities boil down to three schemas, where $\alpha\in\{+,-\}$:

$$\begin{array}{ll}(\forall\alpha) & \mu,x\vDash^\alpha\circ\varphi\iff\forall y\ (xR^\alpha_\circ\implies\mu,y\vDash^\alpha\varphi);\\ (\exists\alpha) & \mu,x\vDash^\alpha\circ\varphi\iff\exists y(xR^\alpha_\circ\text{ and }\mu,y\vDash^\alpha\varphi);\\ (\varnothing\alpha) & \mu,x\vDash^\alpha\circ\varphi\iff x\in v^\alpha(\circ\varphi).\end{array}$$

Then our six modalities use the following schemas:

$$\begin{array}{lll}\Box^v:\ (\forall+),\ (\varnothing-); & \Box^f:\ (\varnothing+),\ (\exists-); & \Box^{vf}:\ (\forall+),\ (\exists-);\\ \Diamond^v:\ (\exists+),\ (\varnothing-); & \Diamond^f:\ (\varnothing+),\ (\forall-); & \Diamond^{vf}:\ (\exists+),\ (\forall-).\end{array}$$

For formulas φ and ψ and a frame $\mathcal{W}$ we write $\varphi\vDash_{\mathcal{W}}\psi$ if for every model μ over $\mathcal{W}$ and element x from $\mu,x\vDash^+\varphi$ implies $\mu,x\vDash^+\psi$. Then we have the following completeness result:

Theorem 0.1 (completeness) *Suppose $\circ\in\{\Box,\Diamond\}$, $\alpha\in\{v,f,vf\}$, then for any formulas φ and ψ we have*

$$\varphi\vdash_{\mathsf{FDE}(X,\circ^\alpha)}\psi\iff\varphi\vDash_{\mathcal{W}}\psi\textit{ for every }\mathsf{FDE}(X,\circ^\alpha)\textit{-frame }\mathcal{W}.$$

Naturally we can add not just one modality but both $\Box$ and $\Diamond$ at the same time. In fact, this is what often happens in the literature. Since there are three basic options for each we get at least nine basic systems denoted $\mathsf{FDE}(X,\Box^\alpha,\Diamond^\beta)$. But we also want to consider extensions of these systems in which some of the modalities and their negated versions are semantically dual (see [11]) in the sense that they use the same accessibility relations. Dunn's [6] gives us a blueprint on how this can be achieved. We summarize these correspondence theory results in the following table: [2]

$$
\begin{aligned}
R^+_\Box = R^+_\Diamond &: \quad \Diamond\varphi \wedge \Box\psi \vdash \Diamond(\varphi \wedge \psi),\ \Box(\varphi \wedge \psi) \vdash \Box\varphi \vee \Diamond\psi;\\
R^-_\Box = R^-_\Box &: \quad \sim\Box\varphi \wedge \sim\Diamond\psi \vdash \sim\Box(\varphi \wedge \psi),\ \sim\Diamond(\varphi \vee \psi) \vdash \sim\Diamond\varphi \vee \sim\Diamond\psi;\\
R^+_\Box = R^+_\Box &: \quad \sim\Box\varphi \wedge \sim\Box\psi \vdash \sim\Box(\varphi \vee \psi),\ \Box\sim(\varphi \vee \psi) \vdash \Box\sim\psi \vee \Box\sim\psi;\\
R^+_\Diamond = R^+_\Diamond &: \quad \Diamond\sim\varphi \wedge \Diamond\sim\psi \vdash \Diamond\sim(\varphi \vee \psi),\ \sim\Diamond(\varphi \vee \psi) \vdash \sim\Diamond\varphi \vee \Diamond\sim\psi;\\
R^+_\Box = R^-_\Diamond &: \quad \Box\sim\varphi \dashv\vdash \sim\Diamond\varphi;\\
R^+_\Diamond = R^-_\Box &: \quad \sim\Box\varphi \dashv\vdash \Diamond\sim\varphi.
\end{aligned}
$$

While most of the systems discussed above fall into this framework modal bilattice logic does not. Yet this situation can be redeemed to an extent. In [15] the authors introduce a generalized system of modal billatice logic (that we call $\mathsf{MBL}^{\mathsf{g}}$ here), in which the single modalitiy of the original system MBL is replaced by two modal operators we denote here as $\Box$ and $\blacksquare$, the conjunction of which gives one the necessity of MBL. One of these operators, $\Box$, is simply a vf-box such that $R^+_\Box = R^-_\Box$; the second, $\blacksquare$, is an f-box, on the one hand, but has the following positive validity condition, on the other:

$$(\forall + tn) \quad \mu, x \models^+ \blacksquare\varphi \iff \forall y\ (xR^+_\blacksquare y \implies \mu, y \not\models^- \varphi);$$

This validity condition schematically resembles that of an impossibility operator [5] but with an in-built toggle between positive and negative validity. We say that $\blacksquare$ is a v_t-impossibility in logic L if the following holds: i) $\blacksquare\varphi \wedge \blacksquare\psi \vdash_L \blacksquare(\varphi \wedge \psi)$; ii) $\varphi \vdash_L \blacksquare dct$; iii) $\sim\varphi \vdash_L \sim\psi / \blacksquare\psi \vdash_L \blacksquare\varphi$.

Then adding a v_t-impossibility corresponds to an unary operator which has $(\forall + tn)$ and $(\varnothing -)$ as its validity conditions in the sense of the completeness result above. It is straightforward to add an operator which is a v_t-impossibility and an f-box simultaneously. This allows as to summarize how different systems discussed above fall into this framework in the following table:

System	Non-modal connectives	Modalities	Semantic dualities
$\mathsf{K}_{\mathsf{FDE}}$	$\varnothing$	$\Box$ is a vf-box	$R^+_\Box = R^-_\Box$
BK	$\to, F$	$\Box$ is a vf-box,	$R^+_\Box = R^-_\Box =$
		$\Diamond$ is a vf-diamond	$R^+_\Diamond = R^-_\Diamond$
$\mathsf{BK}^{\mathsf{FS}}$	$\to, F$	$\Box$ is a vf-box,	$R^+_\Box = R^-_\Box =$
		$\Diamond$ is a vf-diamond	$R^+_\Diamond$
$\mathsf{MBL}^{\mathsf{g}}$	$\otimes, \oplus, \to, F, T, B, N$	$\Box$ is a vf-box, $\blacksquare$ is an f-box	$R^+_\Box = R^-_\Box =$
		and a v_t-impossibility	$R^-_\blacksquare$
$\mathsf{BK}^\Box$	$\to, F$	$\Box$ is a vf-box,	$R^+_\Box = R^-_\Box$
$\mathsf{BK}^{\Box -}$	$\to$	$\Box$ is a vf-box,	$R^+_\Box = R^-_\Box$
$\mathsf{KN4}$	$\Rightarrow$	$\Box$ is a vf-box,	$R^+_\Box = R^-_\Box$

One of the consequences of this is that we now have a uniform axiomatization for all the systems in this table. One might be unsatisfied with this

[2] The result of extending BK with bilattice constants is also considered in [10].

axiomatizations since they characterize sets of formula-formula sequents, yet it is straightforward to pass from this style of axiom systems to the usual Hilbert-style systems in cases, when we do have a weak implication: simply replace $\{\wedge, \vee, \rightarrow\}$-fragment with a Hilbert-style system for classical logic and replace $\vdash$ with $\rightarrow$ in the rest of the axioms and rules. One might be tempted to pass to the usual Hilbert-style systems even in the cases where only strong implication $\Rightarrow$ is in the language, guided by the fact that weak implication can be defined using the strong one. Yet having only strong implication results in a deduction theorem which makes the reduction to theorems very complicated as demonstrated by the calculus for $\mathsf{KN4}$, so perhaps this is as good as it gets for $\mathsf{KN4}$. Another specific question resolved in this paper is the relation between BK and $\mathsf{BK}^{\mathsf{FS}}$: turns out BK can be considered an extension of $\mathsf{BK}^{\mathsf{FS}}$ with just one simple axiom $\Box \sim \varphi \leftrightarrow \Diamond \sim \varphi$.

To reiterate, two directions of extending these results include adding weak and strong intuitionistic implications, which would allow to cover a lot of four-valued modal logics based on variants of Nelson's logic, as well as integrating negative modalities, for which, as $\blacksquare$ above demonstrates, there seem to be a lot more meaningful options than e.g. over intuitionistic logic [5].

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Modeling Forgetting

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Abstract

The usual way of representing information and knowledge in Dynamic Epistemic Logic has one limitation: All information once gained cannot be lost. In this talk I will introduce some concepts of how forgetting or, more generally, the loss of information can be implemented into Action Model Logic(AML). To do so, I introduce *extended model semantics* in which epistemic actions are represented as another type of modality. This will then be used to discuss three concepts of forgetting: "forgetting whether", "forgetting what used to be" and "forgetting what happened." I will only outline how these notions can be implemented into AML.

Keywords: Dynamic epistemic logic, Forgetting, Update logic, Loss of information.

1 Introduction

Agents in all variations of epistemic logic are generally portrayed as being fully aware of all the information they have access to. This is in part because we want to model idealized situations but it also due to the restrictions given by the modal framework used to do so. Forgetting or in general lack of information is an important part of interaction between any type of agents. As such it seems important to remedy this shortcoming by adding the tools necessary to model forgetting to the existing models. The goal of this talk is to give an overview of some ideas of how forgetting can work in a modal logic setting, without going into too much technical detail about how those can be achieved.

In the main part of this talk I will outline three ways of modelling forgetting (or lack of information) for agents of Action Modal Logic(AML). In order to do so I will make use of a semantics that is different from the standard one. My new semantics, called *extended model semantics*, is based on a semantics for Public Announcement Logic given by [7].

2 Extended model semantics for AML

An *epistemic action* α is a triple $\langle K, k, PRE\rangle$ where K is an *action frame* $\langle M^K, R^K\rangle$ such that M^K is a set of *possible actions* and R^K is a set of re-

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lations between these actions for each agent, k is the actual action and PRE is a function from the possible actions to the formula that is its precondition. Possible actions represent all the ways the agents consider an epistemic action might have taken place and $R_a^K kl$ tells us that agent a does not know whether k or l happened given that k happened, e.g. if Ann whispers into Bobs ear, Cathy might consider different possible actions depending on what Ann was saying. The actual action k simply lets us keep track of what actually happened and PRE tells us what needs to be the case for a possible action to be fulfillable at a world. Take a possible action k that states that "Ann tells Bob ϕ". In order for that to be fulfillable ϕ has to be the case, i.e $PRE(k) = \phi$. For two actions α and β which only differ in their actual action (k and l respectively), I use $R_a^\alpha \alpha\beta$ as shorthand for $R_a^K kl$. Here is the axiomatic system:

$$[\alpha]p \leftrightarrow Pre(\alpha) \to p$$
$$[\alpha]\neg\phi \leftrightarrow Pre(\alpha) \to \neg[\alpha]\phi$$
$$[\alpha](\phi \to \psi) \leftrightarrow Pre(\alpha) \to ([\alpha]\phi \to [\alpha]\psi)$$
$$[\alpha]K_a\phi \leftrightarrow Pre(\alpha) \to \bigwedge_{R_a^\alpha \alpha\beta} K_a[\beta]\phi$$

In the standard semantics for AML an epistemic action forces a change in the model by combining worlds with those possible actions whose preconditions they make true, such that $M, w \models [\alpha]\phi$ will be true if and only if $M, w \models PRE(k)$ implies $M^\alpha, \langle w, k\rangle \models \phi$, i.e. ϕ is true at some world $\langle w, k\rangle$ and model M^α that represent w and M after the action happened. In contrast, the idea behind my extended model is to simply extend our starting model with this new changed model. M and M^α will now be part of the same model such that there will be some world v that corresponds to $\langle w, k\rangle$ and $R^\alpha wv$, i.e. there is an α–arrow from w to v for agent a. What is left to do is to make sure that w and v behave accordingly. This is done via normality conditions:

- *A–Functionality*: For any action α: If $M, w \vDash PRE(\alpha)$ then there is a unique v such that $R^\alpha wv$. If $M, w \nvDash PRE(\alpha)$, then there is no such v.
- *A–Invariance*: If $R^\alpha wv$ then for all p: $w \in V(p) \leftrightarrow v \in V(p)$.
- *A–Zig*: If for some agent a and some β such that $R_a^\alpha \alpha\beta$: $R_a ww'$, $R^\alpha wv$ and $R^\beta w'v'$, then $R_a vv'$.
- *A–Zag*; If for some agent a: $R_a vv'$ and $R^\alpha wv$ then there exists an w' and a β such that $R_a ww'$, $R_a^\alpha \alpha\beta$ and $R^\beta w'v'$.

It goes beyond the scope of this talk to go into detail what these normality conditions do. I would just like to point out A–Functionality guarantees that for every w and α there is exactly one w' if w fulfills the condition for α and none otherwise. A–Invariance makes sure w and w' agree on all propositional variables. A–Zig and A–Zag are more complicated but I will note here that they govern how the action and the epistemic modalities interact. Note that given these conditions the following theorem can be shown:

Theorem 2.1 *$M, w \vDash [\alpha]\phi$ holds in extended model semantics if and only if it holds in the standard semantics.*

Proof. Here is a sketch of the proof: Let M^+ be an extended model and M be its epistemic core, i.e. M^+ without the action arrows. If $R^\alpha wv$ is in M^+ and, for every world u in M^+ and every possible action l in α, $M, u \vDash PRE(l)$ according to the extended model semantics iff $M, u \vDash PRE(l)$ according to the standard semantics, then we can show that a world v in M is bisimilar to $\langle w, k \rangle$ in M^α, i.e equivalent. This follows from the normality conditions.

Given this a simple induction on the complexity of the formula suffices to proof the theorem.

□

3 Modeling Forgetting

There are three forms of forgetting I will discuss:

(i) "forgetting whether" formulas hold, i.e. going from knowing some formula ϕ or its negation to no longer having that information,

(ii) "forgetting what used to be", i.e. forgetting which world used to be and

(iii) "forgetting what happened", i.e. forgetting which action took place.

On the first type of forgetting, also called simple forgetting, there has been work done on by [3] and [5] for epistemic logic. I will outline how this idea can be extended to AML in meaningful ways. We take a model and create a lack of information by simply adding those world in that negate the formula that is supposed to be forgotten. Let $[\ddagger p]\phi$ stand for "ϕ is true after forgetting whether the atomic p holds or not". To evaluate this for some model M and world w, we need to create a new model M^p which add worlds to the system in such a way that for every p–world there is a corresponding $\neg p$–world that is in all other ways a copy of that world and vice versa. Without going into too much detail, these wolds have to have incoming and outgoing arrows to the same worlds the original world had in the original model and their copies.

Difficulty arises when one tries to extend this to forgetting of complex formulas. Take the example of forgetting whether $p \wedge q$. There are multiple ways of making this sentence false and so it is not straightforward what exactly a $\neg(p \wedge q)$–world has to look like. For these cases we create minimal cases for which the formula will turn out false. Let w_0 be the original world in the new model $M^{p \wedge q}$. For the two minimal cases in which $p \wedge q$ is false we will now create a world each which fulfills that criterion w_1 in which p is false and w_2 in which q is false. We do not consider the case in which both p and q are false to limit the impact of the forgetting as much as possible. Note that this case can be put on the table again if you additionally forget $p \vee q$. While, I have discussed the notion of "forgetting whether", a weaker notion of "forgetting that" can be defined in a similar way, in which we lose the information about some formula ϕ but not its negation $\neg\phi$.

This account of "forgetting whether" can be directly applied to the extended model semantics given above and will result in a world in which the agent will at no point in the chain of action be able to tell whether a forgotten formula is true or not. However, it does not take into account multiple agents and in the setting of DEL it would represent the"nuclear option" of public forgetting of every agent forgetting and all agents being aware that every agent forgot, i.e. all agents are aware that no one has information about the formula, and so on. A simple way of turning this into forgetting for single agents is by restricting the epistemic relationship on the new model so that all agents that did not forget can only reach the original worlds and the new worlds are only accessible by the forgetting agents. This, however, still keeps all the agents in the loop about what was forgotten and who forgot it. More work still needs to be done to allow for agents to not be aware that other agents have forgotten something and for agents to be able to forget whether other agents know something, i.e. to allow for agent a to forget $K_b\phi$.

Note that this is the only type of forgetting for which we need to add a completely new operation to our system. The other two types will mainly rely on adding a backwards version of the epistemic action operators. It is easy to see that we can do this in a straightforward manner, as we treat epistemic actions as a modality. So let $\widetilde{[\alpha]}\phi$ stand for "before α happened ϕ was the case". So we can define this operation in the standard manner as

Definition 3.1 $\widetilde{[\alpha]}$

$M, v \models \widetilde{[\alpha]}\phi$ iff for all w s.t. $R^\alpha wv$, $M, w \models \phi$.

And add the corresponding version of K as an axiom:

$$\widetilde{[\alpha]}(\phi \to \psi) \to (\widetilde{[\alpha]}\phi \to \widetilde{[\alpha]}\psi).$$

The second type of forgetting I call "forgetting what used to be". The idea here is to model the fact that an agent can no longer tell which action was the last one. This can already be expressed in the semantics given simply because, although we restrict the relations for the epistemic actions to go to at most one world, there is nothing keeping multiple worlds having α–arrows pointing to a single world. For example, let $M, w \models \phi$, $R^\alpha wz$ but also $M, v \models \neg\phi$ and $R^\alpha vz$. Given this now $M, z \models \neg\widetilde{[\alpha]}\phi$ and $M, w \not\models [\alpha]\widetilde{[\alpha]}\phi$. So, in a sense, the fact that ϕ was the case is lost in the step from w to z (given by $[\alpha]$). So, of course, no agent can remember whether ϕ was the case or not and $M, z \not\models K_a\widetilde{[\alpha]}\phi$ directly follows for an arbitrary agent a.

Although this already allows us to model the type of lack of information, it might be worth talking about how to"fix" the system to not allow for such loss of information. This can be done by simply adding another frame restriction to the system that guarantees that the epistemic action relations have to be one-to-one and adding an axiom to the axiomatic system:

$$\widetilde{\langle\alpha\rangle}\phi \to \widetilde{[\alpha]}\phi.$$

The last type of forgetting I want to talk about is "forgetting which action took place". The idea here is that an agent might not be able to tell which world used to be the case before an action happened as well as being even unable to tell what actually happened. Similarly to the second type of forgetting there can be multiple incoming arrows for any world but these arrows are now for different actions. Again take $M, w \models \phi$, $R^\alpha wz$ and $M, v \models \neg\phi$ but now let $R^\beta vz$. We can now use simple action compositionality to define the operator $\widetilde{[\alpha \cup \beta]}$ as "Regardless of whether α or β took place, ϕ used to be." This gives us the following axiom:

$$\widetilde{[\alpha \cup \beta]}\phi \leftrightarrow \widetilde{[\alpha]}\phi \wedge \widetilde{[\beta]}\phi.$$

Given this $M, z \not\models \widetilde{[\alpha \cup \beta]}\phi$ and $M, z \not\models K_a\widetilde{[\alpha \cup \beta]}\phi$ for an arbitrary agent a. So for an agent that does not remember whether α or whether β happened, the knowledge that ϕ used to be the case is lost.

4 Conclusion

I have outlined a new semantics for AML as a foundation for modeling different ways forgetting or loss of information can be modeled in AML. Then I have given my ideas behind three ways of forgetting, both how they can be modeled and how they can be used. These are, however, not all the ways of forgetting. A notion, for example, I have not touched on is that of "forgetting about", i.e. the notion that an agent is no longer aware that a formula ϕ even exists.

For all three cases of forgetting discussed there is still work to be done: For "forgetting whether" I have mentioned that the notion needs to be adapted to be properly incorporated into AML. For "forgetting what happened" it might be interesting to look into other forms of action compositionality and see whether there are fruitful interpretations that correspond to them. Lastly, given the somewhat temporal nature of AML, an approach I would like to explore for all types of forgetting would be to add the option to limit the scope of the forgetting to certain points, i.e the point something was forgotten and a point were something was remembered again.

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A simplicial complex model for epistemic logic

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Abstract

The usual epistemic S5 model for a multi-agent system is based on a Kripke frame, which is a graph whose edges are labeled with agents that do not distinguish between two states. We propose to uncover the higher dimensional information implicit in this structure, by considering a dual, *simplicial complex model*. We show that there is an equivalence of categories between the usual Kripke models and our simplicial models. Thus, desirable properties of Kripke models like soundness and completeness are preserved. What we gain is that we can now study the topological properties of these models, and try to interpret them in terms of knowledge.

Keywords: Epistemic logic, Distributed computing, Simplicial complexes

1 Introduction

The usual Kripke model for epistemic logic **S5** is based on a graph whose nodes are the possible worlds and edges are labeled with agents that do not distinguish between two worlds. We introduce a new kind of model based on *simplicial complexes*. Now, the possible worlds are represented by higher-dimensional simplexes, and the indistinguishability relation corresponds to how the simplexes are glued together. Thus, these models have a topological flavor.

We prove that these simplicial models are very closely related to the usual Kripke models: there is an equivalence of categories between the two structures. This means that both kinds of model actually contain the same information. By going from Kripke models to simplicial models, we uncover the topological structure which is already present, but hidden, in Kripke models. Thus, simplicial models retain the nice properties of Kripke models, such as soundness and completeness w.r.t. (a slightly modified version of) the logic **S5**.

Simplicial models have first been introduced in the context of distributed computing, in order to prove that some distributed *tasks* cannot be solved when processes can crash [6]. Work on knowledge and distributed systems is of course one of the inspirations of the present work [4], especially where connectivity [2,3] is used. In [5], we extend these simplicial models to the setting of dynamic epistemic logic [1,7], and study more in-depth the relationship between knowledge, topology, and distributed computing.

2 A simplicial model for epistemic logic

We describe here the new kind of model for epistemic logic, based on chromatic simplicial complexes.

Syntax. Let AP be a countable set of propositional variables and A a finite set of agents. The language $\mathcal{L}_K$ is generated by the following BNF grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \qquad p \in AP,\ a \in A$$

In the following, we work with $n+1$ agents, and write $A = \{a_0, \ldots, a_n\}$.

Kripke frames. A *Kripke frame* $M = \langle S, \sim \rangle$ over a set A of agents consists of a set of states S and a family of equivalence relations on S, written $\sim_a$ for every $a \in A$. Two states $u, v \in S$ such that $u \sim_a v$ are said to be *indistinguishable* by a. A Kripke frame is *proper* if any two states can be distinguished by at least one agent. Let $M = \langle S, \sim \rangle$ and $N = \langle T, \sim' \rangle$ be two Kripke frames. A *morphism* from M to N is a function f from S to T such that for all $u, v \in S$, for all $a \in A$, $u \sim_a v$ implies $f(u) \sim'_a f(v)$. We write $\mathcal{K}_A$ the category of proper Kripke frames, with morphisms of Kripke frames as arrows.

Simplicial complexes. Given a base set V, a *simplicial* complex C is a family of non-empty finite subsets of V such that for all $X \in C$, $Y \subseteq X$ implies $Y \in C$. We say Y is a *face* of X. Elements of V (identified with singletons) are called *vertices*. Elements of C are *simplexes*, and those which are maximal w.r.t. inclusion are *facets*. The set of vertices of C is noted $\mathcal{V}(C)$, and the set of facets $\mathcal{F}(C)$. The *dimension* of a simplex $X \in C$ is $|X| - 1$. A simplicial complex C is *pure* if all its facets are of the same dimension, n. In this case, we say C is of dimension n. Given the set A of agents (that we will represent as colors), a *chromatic simplicial complex* $\langle C, \chi \rangle$ consists of a simplicial complex C and a coloring map $\chi : \mathcal{V}(C) \to A$, such that for all $X \in C$, all the vertices of X have distinct colors.

Let C and D be two simplicial complexes. A *simplicial map* $f : C \to D$ maps the vertices of C to vertices of D, such that if X is a simplex of C, $f(X)$ is a simplex of D. A *chromatic simplicial map* between two chromatic simplicial complexes is a simplicial map that preserves colors. Let $\mathcal{S}_A$ be the category of pure chromatic simplicial complexes on A.

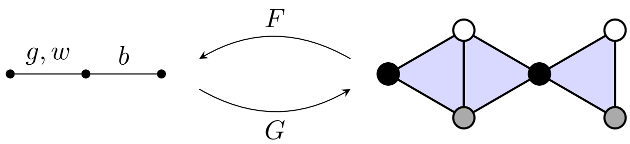
Theorem 2.1 *$\mathcal{S}_A$ and $\mathcal{K}_A$ are equivalent categories.*

Proof (Sketch). We can canonically associate a Kripke frame to a pure chromatic simplicial complex, and vice versa. Let C be a pure chromatic simplicial complex on the set of agents A. We associate the Kripke frame $F(C) = \langle S, \sim \rangle$ with S being the set of facets of C and the equivalence relation $\sim_a$, for all $a \in A$, generated by the relations $X \sim_a Y$ (for X and Y facets of C) if $a \in \chi(X \cap Y)$.

Conversely, consider a Kripke frame $M = \langle S, \sim \rangle$ on the set of agents $A = \{a_0, \ldots, a_n\}$. Intuitively, what we want to do is take one n-simplex $\{v_0^s, \ldots, v_n^s\}$ for each $s \in S$, and glue them together according to the indistiguishability relation. Formally, let $V = \{v_i^s \mid s \in S, 0 \leq i \leq n\}$, and equip it with the equivalence relation R defined by $v_i^s \; R \; v_i^{s'}$ if and only if $s \sim_{a_i} s'$. Then define $G(M)$ whose vertices are the equivalence classes $[v_i^s] \in V/R$, and whose simplexes are of the form $\{[v_0^s], \ldots, [v_n^s]\}$ for $s \in S$, as well as their sub-simplexes. The coloring map is given by $\chi([v_i^s]) = a_i$.

The equivalence is given by the two maps F and G defined above, that we extend to functors between the two categories. $\square$

Example 2.2 The picture below shows a Kripke frame (left) and its associated chromatic simplicial complex (right). The three agents, named b, g, w, are represented as colors black, grey and white on the vertices of the simplicial complex. The three worlds of the Kripke frame correspond to the three triangles (i.e., 2-dimensional simplexes) of the simplicial complex. The two worlds indistinguishable by agent b, are glued along their black vertex; the two worlds indistinguishable by agents g and w are glued along the grey-and-white edge.



We now decorate our simplicial complexes with atomic propositions to get a notion of simplicial model. For technical reasons, we restrict to models where all the atomic propositions are saying something about some local value held by one particular agent. All the examples that we are interested in will fit in that framework. Let $\mathcal{V}$ be some countable set of values, and $AP = \{p_{a,x} \mid a \in A, x \in \mathcal{V}\}$ be the set of *atomic propositions*. Intuitively, $p_{a,x}$ is true if agent a holds the value x. We write AP_a for the atomic propositions concerning agent a.

Kripke models. A *Kripke model* $M = \langle S, \sim, L \rangle$ consists of a Kripke frame $\langle S, \sim \rangle$ and a function $L : S \to \mathscr{P}(AP)$. Intuitively, $L(s)$ is the set of atomic propositions that are true in the state s. A Kripke model is *proper* if the underlying Kripke frame is proper. A Kripke model is *local* if for every agent $a \in A$, $s \sim_a s'$ implies $L(s) \cap AP_a = L(s') \cap AP_a$, i.e., an agent always knows its own values. Let $M = \langle S, \sim, L \rangle$ and $M' = \langle S', \sim', L' \rangle$ be two Kripke models on the same set AP. A *morphism of Kripke models* $f : M \to M'$ is a morphism of the underlying Kripke frames such that $L'(f(s)) = L(s)$ for every state s in S. We write $\mathcal{KM}_{\mathcal{A},AP}$ for the category of local proper Kripke models.

Simplicial models. A *simplicial model* $M = \langle C, \chi, \ell \rangle$ consists of a pure chromatic simplicial complex $\langle C, \chi \rangle$ of dimension n, and a labeling $\ell : \mathcal{V}(C) \to \mathscr{P}(AP)$ that associates to each vertex $v \in \mathcal{V}(C)$ a set of atomic propositions concerning agent $\chi(v)$, i.e., such that $\ell(v) \subseteq AP_{\chi(v)}$. Given a facet $X = \{v_0, \ldots, v_n\} \in C$, we write $\ell(X) = \bigcup_{i=0}^{n} \ell(v_i)$. A *morphism* of simplicial models $f : M \to M'$ is a chromatic simplicial map that preserves the labeling:

$\ell'(f(v)) = \ell(v)$ (and χ). We write $\mathcal{SM}_{A,AP}$ the category of simplicial models over the set of agents A and atomic propositions AP.

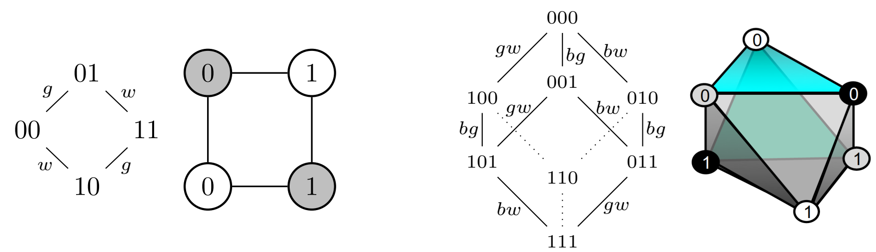
Theorem 2.3 *$\mathcal{SM}_{A,AP}$ and $\mathcal{KM}_{A,AP}$ are equivalent categories.*

Proof (Sketch). We extend the two maps F and G of Theorem 2.1 so that they preserve the labeling ℓ and L of atomic propositions accordingly. □

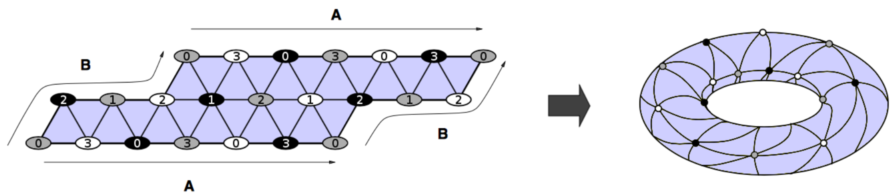
Example 2.4 The figure below shows the so-called binary input complex and its associated Kripke model, for 2 and 3 agents. Each agent gets a binary value 0 or 1, but doesn't know which value has been received by the other agents. So, every possible combination of 0's and 1's is a possible world.

In the Kripke model, the agents are called b, g, w, and the labeling L of the possible worlds is represented as a sequence of values, e.g., 101, representing the values chosen by the agents b, g, w (in that order).

In the simplicial model, the agents are represented as colors (black, grey, and white). The labeling ℓ is represented as a single value in a vertex, e.g., the value 1 in a grey vertex means agent g has chosen value 1. The possible worlds correspond to edges in the 2-agents case, and triangles in the 3-agents case.



Example 2.5 Consider now three agents, and a deck of four cards, $\{0, 1, 2, 3\}$. One card is given to each agent, and the last card is kept hidden. The simplicial model corresponding to that situation is depicted below on the left. The color of vertices indicate the corresponding agent, and the labeling is its card. In the planar drawing, vertices that appear several times with the same color and value should be identified: what we obtain is a triangulated torus.



Keeping that translation in mind, we can reformulate the usual semantics of formulas in Kripke models, in terms of simplicial models.

Definition 2.6 We define the truth of a formula φ in some epistemic state (M, X) with $M = \langle C, \chi, \ell \rangle$ a simplicial model, $X \in \mathcal{F}(C)$ a facet of C and $\varphi \in \mathcal{L}_K(A, AP)$. The satisfaction relation, determining when a formula is true in an epistemic state, is defined as:

$$\begin{array}{lll} M, X \models p & \text{iff} & p \in \ell(X) \\ M, X \models \neg\varphi & \text{iff} & M, X \not\models \varphi \\ M, X \models \varphi \wedge \psi & \text{iff} & M, X \models \varphi \text{ and } M, X \models \psi \\ M, X \models K_a\varphi & \text{iff} & \text{for all } Y \in \mathcal{F}(C), a \in \chi(X \cap Y) \text{ implies } M, Y \models \varphi \end{array}$$

We can show that this definition of truth agrees with the usual one (which we write $\models_{\mathcal{K}}$ to avoid confusion) on the corresponding Kripke model.

Proposition 2.7 *Given a simplicial model M and a facet X, $M, X \models \varphi$ iff $F(M), X \models_{\mathcal{K}} \varphi$. Conversely, given a local proper Kripke model N and state s, $N, s \models_{\mathcal{K}} \varphi$ iff $G(N), G(s) \models \varphi$, where $G(s)$ is the facet $\{v_0^s, \ldots, v_n^s\}$ of $G(N)$.*

Proof. This is straightforward by induction on the formula φ. $\square$

It is well-known that the axiom system **S5** is sound and complete with respect to the class of Kripke models [7]. Since we restrict here to local Kripke models, we need to add the following axiom (or axiom schema, if $\mathcal{V}$ is infinite), saying that every agent knows which values it holds:

$$\mathbf{Loc} \;=\; \bigwedge_{a \in A, x \in \mathcal{V}} K_a(p_{a,x}) \vee K_a(\neg p_{a,x})$$

Corollary 2.8 *The axiom system* **S5** + **Loc** *is sound and complete w.r.t. the class of simplicial models.*

Proof. Adapting the proof of [7] for **S5**, it can be shown that **S5**+**Loc** is sound and complete w.r.t. the class of local proper Kripke models, adapting the usual proof techniques. Then we transpose it to simplicial models using Proposition 2.7. Indeed, suppose a formula φ is true for every local proper Kripke model and any state. Then given a simplicial model and facet (M, X), since by assumption $F(M), X \models_{\mathcal{K}} \varphi$, we also have $M, X \models \varphi$ by Proposition 2.7. So φ is true in every simplicial model. The converse is similar. $\square$

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Proper Display Calculi for Rough Algebras

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Abstract

In this paper, we endow the logics of topological quasi Boolean algebras 5 (tqba5s) and pre-rough algebras with proper multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis and applies the guidelines of the multi-type methodology in the design of display calculi.

Keywords: Rough sets, topological quasi Boolean algebras 5 (tqBa5), pre-rough algebras, multi-type calculi, proper display calculi.

Introduction. Rough algebras and related structures arise in tight connection with formal models of imperfect information [15], and have been investigated for more than twenty years using techniques from universal algebra and algebraic logic, giving rise to a rich theory [1,11,4,16,17]. Recently, sound and complete sequent calculi have been introduced for the logics naturally associated with some of these classes of algebras [16,17]. However, the cut rule in these calculi is not eliminable. In the present paper, we introduce proper display calculi for the two best-known classes related to rough algebras; namely, the so called tqBa5s (acronym for topological quasi Boolean algebras 5, cf. [16]) and pre-rough algebras (cf. Definition 2.3 and 2.6). Our methodology is very akin to the spirit of [1], is driven from algebraic considerations, and is grounded on the general results and insights of the theory of multi-type calculi [7] which has proven effective in endowing many different logical systems (cf. e.g. [13,8,12,9,14,18,6]) with cut-free sequent calculi, and modularly covers also a wide class of axiomatic extensions of given logics [10]. The first contribution of the present paper is an equivalent presentation of tqBa5 and pre-rough algebras, based on so-called *heterogeneous algebras* [3]. Intuitively, heterogeneous algebras are algebras with more than one domain, and operations might span across different domains. The classes of heterogeneous algebras corresponding to tqBa5 and pre-rough algebras have two domains, respectively corresponding to (abstract representations of) *general* sets and *definable* sets of an approximation space, which are assigned distinct *types*. The modal operators capturing the lower and upper definable approximations of a general set are then modeled as heterogeneous maps from the general type to the definable type. The equivalent heterogeneous presentations of tqBa5 and pre-rough algebras come naturally equipped with a multi-type logical language, and are characterized by Hilbert-style axiomatizations which can be readily recognized to be *analytic inductive* (cf. [10, Definition 55]), and hence, by the general theory of analytic proper display calculi, can be effectively captured by proper multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and subformula property. The introduction of these calculi is the second contribution of the present paper.

In ongoing developments of the present results, we show that the present calculi can be modularly extended so as to cover the logics associated with the classes of algebras introduced in [16].

Preliminaries. An algebra $\mathbb{T} = (\mathbb{D}, I)$ is a *topological quasi-Boolean algebra 5* (tqBa5) if it satisfies the following conditions: for all $a, b \in D$,

T1. $\mathbb{D}$ is a De Morgan algebra,	T2. $I(a \wedge b) = Ia \wedge Ib$,	T3. $IIa = Ia$,
T4. $Ia \leq a$,	T5. $I\top = \top$,	T6. $CIa = Ia$,

where $Ca := \neg I \neg a$. By construction, T6 is equivalent to T6′: $ICa = Ca$. A tqBa5 is called a *pre-rough algebra* (pra) if it also satisfies:
P1. $\neg Ia \vee Ia = \top$, P2. $I(a \vee b) = Ia \vee Ib$, P3. $Ia \leq Ib$ and $Ca \leq Cb$ imply $a \leq b$.

Definition 0.1 *The logics of tqBa5s and pras.*

Fix a denumerable set Atprop of propositional variables. The language $\mathcal{L}$ over Atprop is defined recursively as follows:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid IA \mid CA \mid A \wedge A \mid A \vee A,$$

where $p \in \mathsf{Atprop}$. The logic of tqBa5, denoted H.TQBA5, is obtained by adding the following axioms to the logic of De Morgan algebras (cf. [8, Definition 1]):

$$IA \vdash A, \quad IA \vdash IIA, \quad IA \wedge IB \vdash I(A \wedge B), \quad \top \vdash I\top \quad CIA \vdash IA.$$

The logic of pras, denoted H.PRA, is obtained by adding the following axioms and rule to H.TQBA5:

$$I(A \vee B) \vdash IA \vee IB, \quad \top \leq IA \vee \neg IA, \quad \frac{IA \vdash IB \qquad CA \vdash CB}{A \vdash B}$$

The completeness of H.TQBA5 and H.PRA w.r.t. the class of their corresponding algebras can be proven using a routine Lindenbaum-Tarski construction.

Multi-type algebraic representation. Let $\mathbb{T} = (\mathbb{D}, I)$ be a tqBa5s, and let $\mathrm{K} := \{Ia \mid a \in D\}$. Define $\iota : \mathrm{D} \to \mathrm{K}$ and $\gamma : \mathrm{D} \to \mathrm{K}$ by the assignments $a \mapsto Ia$, $a \mapsto Ca$ respectively. T6 and T6′ imply that $\mathrm{K} = \{Ia \mid a \in D\} = \{Ca \mid a \in D\}$. Let $e : \mathrm{K} \hookrightarrow \mathrm{D}$ be the natural embedding. The definition of tqBa5 implies that the operation I (resp. C) is an interior operator (resp. closure operator) on $\mathbb{D}$ seen as a poset. By general order-theoretic facts (cf. [5, Chapter 7]) this means that:

$$\gamma \dashv e \dashv \iota \quad \text{and} \quad \iota(e(\alpha)) = \alpha \quad \text{and} \quad \gamma(e(\alpha)) = \alpha.$$

Let $\mathbb{T}$ be a tqBa5 (resp. pra). The *kernel* of $\mathbb{T}$ is the algebra $\mathbb{K} = (\mathrm{K}, \cup, \cap, -, 1, 0)$ defined as follows: for all $\alpha, \beta \in \mathrm{K}$,

K1. $\alpha \cup \beta := \iota(e(\alpha) \vee e(\beta))$, K2. $0 := \iota(\bot)$, K3. $1 := \iota(\top)$,
K4. $\alpha \cap \beta := \iota(e(\alpha) \wedge e(\beta))$, K5. $-\alpha := \iota\neg e(\alpha)$.

The properties of e, ι and γ imply the following statements:
(1) ι (resp. γ) is a surjective map which preserves meets (resp. joins) and constants;
(2) $e : \mathbb{K} \hookrightarrow \mathbb{T}$ is an injective homomorphism;
(3) If $\mathbb{T}$ is a pra, then ι (resp. γ) also preserves joins (resp. meets).
The following proposition is a consequence of the statements above:

Proposition. If $\mathbb{T}$ is a tqBa5 (resp. pra), then $\mathbb{K}$ is a De Morgan algebra (resp. Boolean algebra).

Definition 0.2 A *heterogeneous tqBa5* (HtqBa5) is a structure $\mathbb{H} := (\mathbb{D}, \mathbb{K}, e, \iota, \gamma)$ such that $\mathbb{D} = (\mathrm{D}, \vee, \wedge, \neg, \top, \bot)$ and $\mathbb{K} = (\mathrm{K}, \cup, \cap, -, 1, 0)$ are De Morgan algebras, $e : \mathbb{K} \hookrightarrow \mathbb{D}$ is a homomorphism, $\iota, \gamma : \mathbb{D} \twoheadrightarrow \mathbb{K}$ are such that $\gamma \dashv e \dashv \iota$ and $\iota(e(\alpha)) = \alpha$ and $\gamma(e(\alpha)) = \alpha$ for all $\alpha \in \mathbb{K}$ (which imply that ι, γ are surjective and e is injective) $\iota(\bot) = 0$ and $\gamma(\top) = 1$, and $\neg e\iota(\neg a) = e\gamma(a)$ for any $a \in \mathrm{D}$.
An HtqBa5 $\mathbb{H}$ as above is a *heterogeneous pre-rough algebra* (Hpra) if $\mathbb{K}$ is a Boolean algebra, $\iota(a \vee b) = \iota(a) \cup \iota(b)$ and $\gamma(a \wedge b) = \gamma(a) \cap \gamma(b)$ for all $a, b \in \mathrm{D}$, and $\iota(a) \leq \iota(b)$ and $\gamma(a) \leq \gamma(b)$ imply $a \leq b$.

$$\begin{tikzcd} \mathbb{K} \arrow[r, hook, "e"'] \arrow[r, phantom, shift left=3, "\top"] \arrow[r, phantom, shift right=4, "\top"] & \mathbb{D} \arrow[l, bend right=50, "\iota"'] \arrow[l, bend left=50, "\gamma"] \end{tikzcd}$$

Given a tqBa5 (resp. pra) $\mathbb{T}$, we let $\mathbb{T}^+ := (\mathbb{D}, \mathbb{K}, e, \iota, \gamma)$ be defined as follows:
(1) $\mathbb{D}$ is the De Morgan algebra (resp. Boolean algebra) reduct of $\mathbb{T}$;
(2) $\mathbb{K}$ is the kernel of $\mathbb{T}$;
(3) e, ι and γ are defined as in the beginning of the present section.
Given an HtqBa5 (resp. Hpra) $\mathbb{H} = (\mathbb{D}, \mathbb{K}, e, \iota, \gamma)$, we let $\mathbb{H}_+ := (\mathbb{D}, I, C)$ where:

(1) $I : \mathbb{D} \to \mathbb{D}$ is defined by the assignment $a \mapsto e(\iota(a))$ for all $a \in \mathbb{D}$;
(2) $C : \mathbb{D} \to \mathbb{D}$ is defined by the assignment $a \mapsto e(\gamma(a))$ for all $a \in \mathbb{D}$.

Proposition. (1) If $\mathbb{T}$ is a tqBa5 (resp. pra), then $\mathbb{T}^+$ is a HtqBa5 (resp. Hpra).
(2) If $\mathbb{H}$ is an HtqBa5 (resp. Hpra), then $\mathbb{H}_+$ is a tqBa5 (resp. pra). Moreover, the kernel of $\mathbb{H}_+$ is isomorphic to $\mathbb{K}$.
(3) For any tqBa5 (resp. pra) $\mathbb{T}$ and any HtqBa5 (resp. Hpra) $\mathbb{H}$:

$$\mathbb{T} \cong (\mathbb{T}^+)_+ \quad \text{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

Proper Display Calculi for H.TQBA5 and H.PRA. The equivalent presentation of tqBa5s and pras in terms of their heterogeneous counterparts provides natural algebraic semantics for the languages of the calculi D.TQBA5 and D.PRA, consisting of types D and K, where the heterogeneous connectives $\circ, \blacksquare, \blacklozenge$ are respectively interpreted as e, ι, γ in heterogeneous algebras. This language includes both logical and structural connectives. The structural counterpart of a given logical connective is denoted by decorating that logical connective with ˆ (resp. ˇ and ˜). The order-theoretic motivation underlying this notation is discussed in [8].

$$\mathsf{D}\left\{\begin{array}{l} A ::= p \mid \top \mid \bot \mid \circ\,\alpha \mid \neg A \mid A \wedge A \mid A \vee A \\ X ::= A \mid \check{\bot} \mid \hat{\top} \mid \tilde{\circ}\,\Gamma \mid (\check{\Box}\,\Gamma) \mid (\hat{\Diamond}\,\Gamma) \mid \tilde{\neg}\,X \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid X \mathbin{\hat{\succ\!\!-}} X \mid X \mathbin{\check{\rightarrow}} X \end{array}\right.$$

$$\mathsf{K}\left\{\begin{array}{l} \alpha ::= \blacksquare A \mid \blacklozenge A \mid 1 \mid 0 \mid -\alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \\ \Gamma ::= \check{\blacksquare}\,X\,(\tilde{\blacksquare}X) \mid \hat{\blacklozenge}\,X\,(\tilde{\blacklozenge}X) \mid \hat{1} \mid \check{0} \mid \tilde{-}\,\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid \Gamma \mathbin{\hat{\supset\!\!-}} \Gamma \mid \Gamma \mathbin{\check{-\!\!\supset}} \Gamma \end{array}\right.$$

where $\tilde{\blacksquare}X$ (resp. $\tilde{\blacklozenge}X$) replaces $\check{\blacksquare}\,X$ (resp. $\hat{\blacklozenge}\,X$) when moving from the language of D.TQBA5 to the language of the D.PRA, and $\check{\Box}\,\Gamma$ (resp. $\hat{\Diamond}\,\Gamma$) which only occurs in the language of the D.PRA is interpreted in heterogeneous pras as the right (resp. left) adjoint of ι (resp. γ) The pure D- and K-type rules of the proper multi-type display calculus D.TQBA5 are the same rules of the proper display calculus for De Morgan logic (cf. [8, Section 5.2]), therefore we omit them. The introduction rules for the heterogeneous connectives are standard and likewise omitted. D.TQBA5 is characterized by the following multi-type structural rules:

$$ad_{\mathsf{DK}}\,\frac{\tilde{\circ}\,\Gamma \vdash Y}{\Gamma \vdash \check{\blacksquare}\,Y} \qquad \frac{\Pi \vdash \tilde{\circ}\,Y}{\hat{\blacklozenge}\,\Pi \vdash Y}\,ad_{\mathsf{DK_C}}$$

$$\tilde{\circ}\,\hat{1}\,\frac{\hat{\top} \vdash Y}{\tilde{\circ}\,\hat{1} \vdash Y} \qquad \frac{\Gamma \vdash \check{\blacksquare}\,\check{\bot}}{\Gamma \vdash \check{0}}\,\check{\blacksquare}\,\hat{\top} \qquad \frac{\tilde{\circ}\,\Gamma \vdash \tilde{\circ}\,\Delta}{\Gamma \vdash \Delta}\,\tilde{\circ} \qquad \frac{\hat{\blacklozenge}\,X \vdash \check{\blacksquare}\,Y}{X \vdash Y}\,\hat{\blacklozenge}\,\check{\blacksquare}$$

$$CI\,\frac{\tilde{\circ}\,\hat{\blacklozenge}\,\tilde{\neg}\,X \vdash Y}{\tilde{\neg}\,\tilde{\circ}\,\check{\blacksquare}\,X \vdash Y} \qquad \frac{X \vdash \tilde{\circ}\,\check{\blacksquare}\,\tilde{\neg}\,Y}{X \vdash \tilde{\neg}\,\tilde{\circ}\,\hat{\blacklozenge}\,Y}\,IC \qquad \frac{X \vdash \tilde{\circ}\,\tilde{-}\,\Gamma}{X \vdash \tilde{\neg}\,\tilde{\circ}\,\Gamma}\,\tilde{\circ}\,\tilde{-}$$

The calculus D.PRA is obtained by adding the following rules to D.TQBA5:

$$adj_{\mathsf{K_I}}\ \frac{\tilde{\blacksquare} X \vdash \Gamma}{\overline{X \vdash \check{\Box}\Gamma}} \qquad \frac{\Gamma \vdash \tilde{\blacklozenge} X}{\overline{\hat{\Diamond}\Gamma \vdash X}}\ adj_{\mathsf{K_I}}$$

$$exm\ \frac{\Gamma \,\hat{\cap}\, \Delta \vdash \Lambda}{\Delta \vdash \tilde{-}\Gamma \,\check{\cup}\, \Lambda} \qquad \frac{\tilde{\blacksquare} X \vdash \tilde{\blacksquare} Y \qquad \tilde{\blacklozenge} X \vdash \tilde{\blacklozenge} Y}{X \vdash Y}\ ro$$

Properties. The background theory of multi-type calculi [7,10,13,2] guarantees that:

1. D.TQBA5 (resp. D.PRA) is sound w.r.t. perfect HtqBa5s (resp. Hpras);
2. D.TQBA5 (D.PRA) is complete w.r.t. the class of tqBas (pras);
3. D.TQBA5 (D.PRA) is a conservative extension of H.TQBA (H.PRA);
4. D.TQBA5 (D.PRA) enjoys cut-elimination and subformula property.

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Non-finitely Axiomatisable Modal Products with Infinite Canonical Axiomatisations

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Abstract

We give the first examples of products of finitely axiomatisable modal logics that are not finitely axiomatisable but axiomatisable by explicit infinite sets of canonical (sometimes even Sahlqvist) axioms. In particular, we study here modal products with **Diff**, the propositional unimodal logic of the difference operator. We show that the 2D product logic $\mathbf{Diff} \times \mathbf{Diff}$ is non-finitely axiomatisable, but can be axiomatised by infinitely many Sahlqvist axioms. We also show that its 'square' version $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ (the modal counterpart of two-variable substitution and equality free first-order logic with counting to 2) is non-finitely axiomatisable over $\mathbf{Diff} \times \mathbf{Diff}$, but can be axiomatised by adding infinitely many canonical axioms.

Keywords: products of modal logics, non-finite axiomatisability, canonical and Sahlqvist axiomatisations, elsewhere quantifiers

1 Introduction

Ever since their introduction [20,22,5], products of modal logics —propositional multimodal logics determined by classes of product frames— have been extentensively studied, see [4] for a comprehensive exposition and further references.

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In this paper we consider the problem of finding explicit infinite 'nice' axiomatisations for non-finitely axiomatisable two-dimensional modal product logics. By being 'nice' here we mean formulas to which both the canonicity and first-order correspondence properties of Sahlqvist formulas apply.

It is well-known that the 2D modal product logic **S5**$\times$**S5** (the modal counterpart of two-variable substitution and equality free first-order logic) has a finite axiomatisation with Sahlqvist axioms, expressing that both modalities are **S5** and they commute [8]. On the other hand, for $n > 2$ the n-dimensional product logic $\mathbf{S5}^n$ is not only non-finitely axiomatisable [11], but every axiomatisation for it must contain infinitely many non-canonical axioms [10,2], even if it is itself a canonical and r.e. [8] (though undecidable [17]) logic. There are also known examples of recursively enumerable (even decidable) 2D products of finitely axiomatisable modal logics that are not finitely axiomatisable, such as **K4.3**$\times$**S5** [16]. However, so far no canonical axiomatisations for non-finitely axiomatisable products of finitely axiomatisable logics have been known.

Instead of **S5** (the modal logic of all equivalence relations), here we study modal products with the finitely axiomatisable [21] logic **Diff** of all non-equality frames $(W, \neq)$. An arbitrary frame for **Diff** is a *pseudo-equivalence* relation: its equivalence classes might contain both reflexive and irreflexive points. It is easy to see that, unlike equivalence relations, the class of pseudo-equivalence relations is not Horn-definable. Therefore, the general theorem of Gabbay and Shehtman [5] about axiomatising 2D products of Horn-definable logics by their commutator does not apply to **Diff**$\times$**Diff**. However, as pseudo-equivalence relations form an elementary class, it does follow by general results on product logics [5,15] that **Diff**$\times$**Diff** is canonical and r.e.

We show that the 2D product logic **Diff**$\times$**Diff** is non-finitely axiomatisable, but can be axiomatised by infinitely many Sahlqvist axioms. We also show that its 'square' version **Diff**$\times^{sq}$**Diff** (the modal counterpart of two-variable substitution and equality free first-order logic with counting to 2) is non-finitely axiomatisable over **Diff**$\times$**Diff**, but can be axiomatised by adding infinitely many axioms that are generalised Sahlqvist à la [6]. This way we give the first examples of products of finitely axiomatisable modal logics that are not finitely axiomatisable but axiomatisable by explicit infinite sets of canonical axioms. Moreover, each of our axioms has a first-order correspondent. Our logics are also counterexamples for the dichotomy described in [13].

2 Definitions

In what follows we assume that the reader is familiar with the basics of possible world semantics for propositional multimodal logics, Sahlqvist formulas and canonicity (see, e.g., [1,3]). Here we introduce some of the used notions only.

To begin with, we use the bimodal language $\mathcal{L}_2$ whose formulas are built up from propositional variables using the Booleans and the unary modal operators $\Box_0$, $\Box_1$, and $\Diamond_0$, $\Diamond_1$. By a (*normal bi*)*modal logic* we mean any set L of bimodal formulas that contains all propositional tautologies and the **K**-formulas for both modalities, and is closed under the rules of Substitution, Modus Ponens, and Necessitation for both modalities. A recursive set Σ of formulas *axiomatises* L if L is the smallest modal logic containing Σ. (Clearly, only r.e. logics can be axiomatised in this sense.) Bimodal formulas are evaluated in Kripke models over 2-frames (W, R_0, R_1), where R_0 and R_1 are binary relations on some nonempty set W.

Given unimodal Kripke frames $\mathfrak{F}_0 = (W_0, R_0)$ and $\mathfrak{F}_1 = (W_1, R_1)$, their *product* is defined to be the special 2-frame

$$\mathfrak{F}_0 \times \mathfrak{F}_1 = (W_0 \times W_1, \overline{R}_0, \overline{R}_1),$$

where $W_0 \times W_1$ is the Cartesian product of W_0 and W_1 and, for all $x, x' \in W_0$, $y, y' \in W_1$,

$$\begin{array}{lll} (x,y)\overline{R}_0(x',y') & \text{iff} & xR_0x' \text{ and } y = y', \\ (x,y)\overline{R}_1(x',y') & \text{iff} & yR_1y' \text{ and } x = x'. \end{array}$$

For $i = 0, 1$, let L_i be a Kripke complete unimodal logic in the language with $\Box_i$ and $\Diamond_i$. The *product* of L_0 and L_1 is defined as the bimodal logic

$$L_0 \times L_1 = \{\varphi \in \mathcal{L}_2 : \varphi \text{ is valid in } \mathfrak{F}_0 \times \mathfrak{F}_1 \text{ for any frame } \mathfrak{F}_i \text{ for } L_i,\ i = 0, 1\}.$$

In particular,

$$\mathbf{Diff} \times \mathbf{Diff} = \{\varphi \in \mathcal{L}_2 : \varphi \text{ is valid in } \mathfrak{F} \times \mathfrak{G} \text{ for any frames } \mathfrak{F}, \mathfrak{G} \text{ for } \mathbf{Diff}\}.$$

It is easy to see that

$$\mathbf{Diff} \times \mathbf{Diff} = \{\varphi \in \mathcal{L}_2 : \varphi \text{ is valid in } (W_0, \neq) \times (W_1, \neq) \text{ for any non-empty sets } W_0,\ W_1\}.$$

We also define the 'square' version as

$$\mathbf{Diff} \times^{sq} \mathbf{Diff} = \{\varphi \in \mathcal{L}_2 : \varphi \text{ is valid in } (W, \neq) \times (W, \neq) \text{ for any non-empty set } W\}.$$

Note that it is easy to see that $\mathbf{S5} \times \mathbf{S5} = \mathbf{S5} \times^{sq} \mathbf{S5}$, but in case of **Diff** the two versions turn out to be very different, see Theorem 3.4 (i) below.

It is not hard to show by reductions to two-variable first-order logic with counting that both $\mathbf{Diff} \times \mathbf{Diff}$ and $\mathbf{Diff} \times^{sq} \mathbf{Diff}$ are decidable (and so r.e.) [7]. Therefore, it is decidable whether a finite 2-frame is a frame for any of these logics (by using Yankov-Fine formulas and the decision procedure for validity).

3 Results

Theorem 3.1 *Let L be any bimodal logic such that*

- *L contains $\mathbf{K} \times \mathbf{Diff}$, and*
- *$(W, W \times W) \times (\omega, \neq)$ is a frame for L, for some infinite or arbitrarily large finite W.*

Then L is not axiomatisable using finitely many propositional variables.

Corollary 3.2 *For any Kripke complete logic L with* $\mathbf{K} \subseteq L \subseteq \mathbf{S5}$*,* $L\times\mathbf{Diff}$ *is not axiomatisable using finitely many propositional variables. Thus,* $\mathbf{Diff}\times\mathbf{Diff}$ *is non-finitely axiomatisable.*

Theorem 3.1 generalises some results of [14]. The theorem is shown by giving two infinite sequences $\mathfrak{F}_n$ and $\mathfrak{G}_n$ $(n < \omega)$ of 2-frames such that (i) $\mathfrak{F}_n$ is not a frame for $\mathbf{K}\times\mathbf{Diff}$, (ii) $\mathfrak{G}_n$ is a p-morphic image of $(\omega, \omega\times\omega)\times(\omega, \neq)$ or $(W, W\times W)\times(\omega, \neq)$ for large enough finite W, and (iii) if $n \geq 2^m$ then $\mathfrak{F}_n$ and $\mathfrak{G}_n$ cannot be distinguished by any m-variable bimodal formula.

Theorem 3.3 *There is an explicit infinite axiomatisation for* $\mathbf{Diff}\times\mathbf{Diff}$ *consisting of Sahlqvist formulas.*

The proof is of the following pattern: The explicit axioms describe countable *forbidden* 2*-frames*: frames that are not p-morphic images of 2D-products of non-equality frames. As each axiom is a Sahlqvist formula, the logic they axiomatise is Kripke complete and has the countable frame property. As the axioms are valid in 2D-products of non-equality frames, they in fact axiomatise $\mathbf{Diff}\times\mathbf{Diff}$.

Using similar methods, we also show:

Theorem 3.4 (i) $\mathbf{Diff}\times^{sq}\mathbf{Diff}$ *is not axiomatisable over* $\mathbf{Diff}\times\mathbf{Diff}$ *by any set of axioms using finitely many propositional variables,* (ii) *but it can be axiomatised by adding infinitely many generalised Sahlqvist formulas* [6] *to* $\mathbf{Diff}\times\mathbf{Diff}$.

4 Discussion

Here are some related loose ends:

(1) Hirsch and Hodkinson [9] give an explicit infinite axiomatisation for (the algebraic counterpart of) $\mathbf{S5}^n$, for any $n < \omega$. The axioms are obtained by first expressing 'universally' the winning strategy for $\exists$ in a two-player 'representation' game, and then turning these 'universal expressions' to modal formulas by using that there is a universal modality in $\mathbf{S5}^n$-frames. By the negative results of [10,2], infinitely many of these axioms cannot be Sahlqvist.

It is easy to see that the method of [9] can also be used to give an explicit infinite axiomatisation for $\mathbf{Diff}\times\mathbf{Diff}$. Are the obtained axioms Sahlqvist/canonical?

(2) We do not know whether Theorem 3.4 (ii) holds with Sahlqvist formulas in place of generalised Sahlqvist. The method of Kikot [12] testing for Sahlqvist axiomatisability does not seem to apply.

(3) $\mathbf{Diff}\times^{sq}\mathbf{Diff}$ is the modal counterpart of a fragment of two-variable first-order logic with counting to 2. It is shown in [18] that the satisfiability problem of this fragment is NExpTime-complete. Pratt-Hartmann [19] gives another proof of this, connecting type-structures of the fragment to integer programming problems. As our proof of Theorem 3.4 (ii) also has some integer programming flavour, it would be interesting to connect our methods to those of [19]. Perhaps such a connection could simplify our (quite complex) axioms?

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Categories of coalgebras for modal extensions of Łukasiewicz logic

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Abstract

The category of complete and completely distributive Boolean algebras with complete operators is dual to the category of frames. We lift this duality to the category of complete and completely distributive MV-algebras with complete operators.

Keywords: coalgebras, many-valued logics, Łukasiewicz logic, MV-algebras, modal logic

1 Introduction

Modal extensions of Łukasiewicz logics have been investigated for their theoretical aspects [1,2,6,8,11] and their applications [4,7,10]. The links between relational semantics and modal extensions of finitely-valued Łukasiewicz logics can be efficiently approached algebraically by adapting the corresponding tools from Boolean modal logic [6,9]. Modal extensions of infinitely-valued Łukasiewicz logics are much difficult to deal with, as representation results for their algebraic counterparts are missing. For instance, the known axiomatizations of these logics involve deduction rules with an infinite number of premises [1,6].

The purpose of this note is to exhibit a result about modal extensions of infinite-valued Łukasiewicz logic that is deeply algebraic in nature, showing that algebras can nevertheless be useful to investigate this class of logics.

Let CBAO be the category of complete and completely distributive Boolean algebras with complete operators, and $\mathcal{P}\mathsf{CO}$ be the category of $\mathcal{P}$-coalgebras, where $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ is the power set functor. It was proved in [12] that these categories are dually equivalent. In this note, we lift this result at the level of $\mathcal{P}_{\mathbb{N}}$-coalgebras, for a a suitable power set functor $\mathcal{P}_{\mathbb{N}}\colon \mathsf{Set}_{\mathbb{N}} \to \mathsf{Set}_{\mathbb{N}}$ defined on

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the category of $\mathbb{N}$-fuzzy sets, where $\mathbb{N}$ is the set of positive integers ordered by divisibility. The algebraic counterpart of this duality is the category of complete and completely distributive MV-algebras with complete operators, and complete homomorphisms as arrows. By restricting this duality to the full subcategory of $\mathcal{P}_{\mathrm{div}(n)}$-coalgebras (where $\mathrm{div}(n)$ denotes the poset of divisors of any $n > 0$) and complete and completely distributive MV_n-algebras with complete operators, we obtain a dual equivalence for the category of Ł_n-frames that has already proved to provide a rich relational semantics for modal extensions of $n+1$-valued Łukasiewicz logics [6,11].

Let $\preceq$ be the partial order over the nonnegative integers $\mathbb{N}$ defined by $a \preceq b$ if b is a divisor of a. The partially ordered set $\mathbf{N} = (\mathbb{N}, \preceq)$ is a complete lattice with 1 and 0 as top and bottom elements, respectively. If $X \subseteq \mathbb{N}$, then $\curlyvee X$ is the greatest common divisor of the elements of X, and $\curlywedge X$ is their least common multiple if X is finite, and 0 otherwise. For every positive integer n, we denote by $\mathrm{div}(n)$ the sublattice of $\mathbf{N}$ made of the divisors of n. We denote by Ł_0 the standard MV-algebra $([0,1], \to, \neg, 0)$, where $x \to y = \min\{1, 1 - x + y\}$ and $\neg x = 1 - x$ for every x, y in $[0,1]$. For every $n > 0$, we denote by Ł_n the subalgebra $\{0, 1/n, \ldots, (n-1)/n, 1\}$ of Ł_0. We reserve the notation inf and sup for the *infimum* and *supremum* computed in $[0,1]$ relatively to the natural order.

2 Relational semantics for modal extensions of Łukasiewicz logics

Let Form be the set of formulas constructed on the language $\{\neg, \Box, \to, 0\}$ from a countable set Prop of propositional variables in the usual way. We use the customary abbreviations in Łukasiewicz logic: we write $p \oplus q$ for $\neg p \to q$, $p \odot q$ for $\neg(\neg p \oplus \neg q)$, $x \vee y$ for $(y \odot \neg x) \oplus x$, $x \wedge y$ for $(y \oplus \neg x) \odot x$, and 0 for $\neg 1$.

Given a complete lattice $\mathbf{L} = (L, \leq)$, Goguen introduced in [5] the category of $\mathbf{L}$-fuzzy sets whose objects are the ordered pairs (X, v) where $X \in \mathsf{Set}$ and $v \colon X \to L$, and whose arrows $f \colon (X, v) \to (X', v')$ are maps $f \colon X \to X'$ such that $v' \circ f \geq v$.

For instance, every $X \in \{\text{Ł}_n \mid n \geq 0\}$ is considered as an $\mathbf{N}$-fuzzy set $\mathbf{X} = (X, v_X)$ by defining $v_X \colon X \to \mathbb{N}$ as

$$v_X(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ n/\gcd(i,n) & \text{if } x = \frac{i}{n} \in \mathbb{Q}. \end{cases}$$

We denote $(\text{Ł}_n, v_{\text{Ł}_n})$ by $\mathbf{Ł_n}$, and $([0,1], v_{[0,1]})$ by $[\mathbf{0}, \mathbf{1}]$.

For any complete lattice $\mathbf{L}$, the power set functor $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$ can be naturally lifted into $\mathcal{P}_{\mathbf{L}} \colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}_{\mathbf{L}}$ by setting $\mathcal{P}_{\mathbf{L}}(X, v) = (\mathcal{P}(X), v_{\mathcal{P}})$ where $v_{\mathcal{P}}$ is defined by $v_{\mathcal{P}}(Y) = \bigwedge v(Y)$ for any $Y \in \mathcal{P}(X)$, and by setting $\mathcal{P}_{\mathbf{L}}(f)(Y) = f(Y)$ for every $f \colon (X, v) \to (X', v')$, and every $Y \subseteq X$. We denote by $\mathcal{P}_{\mathbf{L}}\mathsf{CO}$ the category of $\mathcal{P}_{\mathbf{L}}$-coalgebras. For instance, for $\mathbf{L} = \mathbf{N}$ or $\mathbf{L} = \mathrm{div}(n)$, and for any $(X, v) \in \mathsf{Set}_{\mathbf{L}}$, a map $R \colon X \to \mathcal{P}(X)$ is a $\mathcal{P}_{\mathbf{L}}$-coalgebra if and only if $\curlywedge v(Rx) \geq v(x)$ for every $x \in X$, that is, if $v(y)$ is a divisor of $v(x)$ for every

y in Rx and every x in X. In particular, it is easily seen that the category of $\mathcal{P}_{\mathrm{div}(n)}$-coalgebras is equivalent to the category of Ł$_n$-frames considered in [6,11], and provides a natural semantics for Form.

Definition 2.1 Let $\mathbf{L} \in \{\mathbb{N}, \mathrm{div}(n)\}$ and $(X, v) \in \mathsf{Set}_{\mathbf{L}}$. An $\mathbf{L}$-*valuation* on (X, v) is a map $\mathrm{Val}\colon X \times \mathsf{Prop} \to [0,1]$ such that $\mathrm{Val}(-,p)\colon (X,v) \to ([0,1], v_{[0,1]})$ is a $\mathsf{Set}_{\mathbf{L}}$-morphism for every $p \in \mathsf{Prop}$.

If R is a $\mathcal{P}_{\mathbf{L}}$-coalgebra on (X, v), an $\mathbf{L}$-valuation Val on (X, v) is extended to $X \times \mathsf{Form}$ using Łukasiewicz interpretation of the connectors $\to, \neg, 0$, and the rule

$$\mathrm{Val}(x, \Box\phi) = \inf\{\mathrm{Val}(y, \phi) \mid y \in Rx\},$$

for any $x \in X$, and $\phi \in \mathsf{Form}$.

Informally, for $\mathbf{L} \in \{\mathbb{N}, \mathrm{div}(n)\}$, a $\mathsf{Set}_{\mathbf{L}}$-coalgebra can be considered as a relational structure in which modal formulas are evaluated using valuations in $[0,1]$, with some prescriptions on the set of truth values in each world of the structure.

3 Complete and completely distributive MV-algebras with complete operator

The structure of complete and completely distributive MV-algebras is well understood [3]. Recall that the algebras Ł$_n$ (for $n \geq 0$) are the only complete totally ordered MV-algebras. For every MV-algebra $\mathbf{A}$, we denote by $\mathfrak{B}(\mathbf{A})$ its *Boolean skeleton*, that is, we have $\mathfrak{B}(\mathbf{A}) = \{a \in A \mid a \oplus a = a\}$

Theorem 3.1 ([3]) *For every MV-algebra* $\mathbf{A}$*, the following conditions are equivalent.*

(i) $\mathbf{A}$ *is complete and completely distributive.*

(ii) $\mathbf{A}$ *is a direct product of complete and totally ordered MV-algebras.*

(ii) $\mathbf{A}$ *is complete and the Boolean algebra* $\mathfrak{B}(\mathbf{A})$ *is atomic.*

Moreover, if $\mathbf{A}$ *satisfies one of the above conditions, then* $\mathbf{A}$ *is isomorphic to the direct product of the complete MV-chains* $(a] := \{x \in A \mid x \leq a\}$*, for* a *in the set of atoms of* $\mathfrak{B}(\mathbf{A})$*.*

If $\mathbf{A}$ is a complete and completely distributive MV-algebra and if a is an atom of $\mathfrak{B}(\mathbf{A})$, we denote by i_a the unique MV-embedding $i_a\colon (a] \to$ Ł$_0$.

Theorem 3.1 suggests to dualize $\mathcal{P}_{\mathbb{N}}$-coalgebras through the construction of complex algebras as follows.

Definition 3.2 The *complex algebra* associated with a $\mathcal{P}_{\mathbb{N}}$-coalgebra $R\colon \mathbf{X} \to \mathcal{P}_{\mathbb{N}}(\mathbf{X})$ is the algebra $\mathbf{R}_* = (\mathrm{Hom}(\mathbf{X}, [\mathbf{0}, \mathbf{1}]), \Box_R)$, where $\mathrm{Hom}(\mathbf{X}, [\mathbf{0}, \mathbf{1}])$ has the MV-algebra structure inherited from $[0,1]^X$, and $\Box_R$ is the map defined on $\mathrm{Hom}(\mathbf{X}, [\mathbf{0}, \mathbf{1}])$ as

$$(\Box_R f)(x) = \inf\{f(y) \mid y \in Rx\}.$$

If $R\colon \mathbf{X} \to \mathcal{P}_{\mathbb{N}}(\mathbf{X})$ and $R'\colon \mathbf{Y} \to \mathcal{P}_{\mathbb{N}}(\mathbf{Y})$ are two $\mathcal{P}_{\mathbb{N}}$-coalgebras, and if $g\colon R \to R'$ is a $\mathcal{P}_{\mathbb{N}}$-coalgebra morphism, we denote by g_* the map defined on

$\mathbf{R'}_*$ by

$$g_*(f)(x) = f(g(x)).$$

Lemma 3.3 *Let $R\colon \mathbf{X} \to \mathcal{P}_{\mathbb{N}}(\mathbf{X})$ and $R'\colon \mathbf{Y} \to \mathcal{P}_{\mathbb{N}}(\mathbf{Y})$ be two $\mathcal{P}_{\mathbb{N}}$-coalgebras, and $g\colon R \to R'$ be a $\mathcal{P}_{\mathbb{N}}$-coalgebra morphism.*

(A) The MV-algebra reduct of $\mathbf{R}_*$ is a complete and completely distributive MV-algebra.

(B) For all $\star \in \{\oplus, \odot\}$, the algebra $\mathbf{R}_*$ satisfies the equations

$$\Box(x \star x) = \Box x \star \Box x, \tag{1}$$
$$\Box 1 = 1 \tag{2}$$

(C) The map $\Box_R$ is valued in $\mathrm{Hom}(\mathbf{X}, [\mathbf{0}, \mathbf{1}])$ *and is complete,* i.e., *for any family $\{f_\alpha \mid \alpha \in A\}$ of elements of* $\mathrm{Hom}(\mathbf{X}, [\mathbf{0}, \mathbf{1}])$*, it holds*

$$\Box_R(\inf\{f_\alpha \mid \alpha \in A\}) = \inf\{\Box_R f_\alpha \mid \alpha \in A\}. \tag{3}$$

(D) The map g_* is valued in $\mathbf{R}_*$ and is a complete MV-algebra homomorphism that satisfies $g_*(\Box_{R'} f) = \Box_R g_*(f)$ for every $f \in \mathbf{R'}_*$.

Definition 3.4 We denote by CMVO the category of complete and completely distributive MV-algebras with complete operators.

The objects of CMVO are the complete and completely distributive MV-algebras equipped with a unary operation $\Box$ that satisfies equations (1) – (2), and has property (3).

An arrow $f\colon \mathbf{A} \to \mathbf{B}$ of CMVO is a complete MV-algebra homomorphism that satisfies $f(\Box a) = \Box f(a)$ for any $a \in A$.

It can easily be proven by Lemma 3.3, that $-_*$ is a functor from the category of $\mathcal{P}_{\mathbb{N}}$-coalgebras to the category CMVO.

4 Atomic coalgebras

As stated in the next proposition, a complete modal operator on a complete and completely distributive MV-algebra is fully determined by its restriction to the set of atoms of its Boolean skeleton.

Proposition 4.1 *Let $(\mathbf{A}, \Box)$ and $(\mathbf{A}, \Box')$ be two objects of* CMVO *with a common MV-reduct $\mathbf{A}$. If $\Box'$ and $\Box$ coincide on* $\mathrm{Atom}(\mathfrak{B}(\mathbf{A}))$ *then $\Box' = \Box$.*

Proposition 4.1 suggests the following construction.

Lemma 4.2 *Let $\mathbf{A} \in \mathsf{CMVO}$. If the maps $v_{\mathbf{A}}\colon \mathrm{Atom}(\mathfrak{B}(\mathbf{A})) \to \mathbb{N}$ and $R_{\mathbf{A}}\colon \mathrm{Atom}(\mathfrak{B}(\mathbf{A})) \to \mathcal{P}(\mathrm{Atom}(\mathfrak{B}(\mathbf{A})))$ are defined as*

$$v_{\mathbf{A}}(a) = n \quad \text{if} \quad (a] \cong L_n, \tag{4}$$
$$R_{\mathbf{A}}(a) = \{b \in \mathrm{Atom}(\mathfrak{B}(\mathbf{A})) \mid a \leq \diamond b\}, \tag{5}$$

then $R_{\mathbf{A}}$ is a $\mathcal{P}_{\mathbb{N}}$-coalgebra over $\mathbf{A}^* := (\mathrm{Atom}(\mathfrak{B}(\mathbf{A})), v_{\mathbf{A}})$.

Definition 4.3 The *atomic coalgebra* associated with $\mathbf{A} \in \mathsf{CMVO}$ is the $\mathcal{P}_{\mathbb{N}}$-coalgebra $R_{\mathbf{A}} \colon \mathbf{A}^* \to \mathcal{P}_{\mathbb{N}}(\mathbf{A}^*)$ defined as in Lemma 4.2.

Informally, the atomic coalgebra associated with $\mathbf{A} \in \mathsf{CMVO}$ is obtained by turning the atomic frame of $\mathfrak{B}(\mathbf{A})$ into a $\mathcal{P}_{\mathbb{N}}$-colagebra.

We extend the definition of R_{-} on arrows of CMVO as follows.

Proposition 4.4 *Let* $\mathbf{A}, \mathbf{B} \in \mathsf{CMVO}$ *and* $f \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$*. The mapping* R_f *defined on* $\mathrm{Atom}(\mathfrak{B}(\mathbf{B}))$ *by*

$$R_f(b) = \wedge\{a \in \mathfrak{B}(\mathbf{A}) \mid b \leq f(a)\}$$

is an $\mathcal{P}_{\mathbb{N}}$*-coalgebra morphsim* $R_f \colon R_{\mathbf{B}} \to R_{\mathbf{A}}$*.*

5 A dual equivalence

The main result of this note is a dual equivalence between CMVO and the category $\mathcal{P}_{\mathbb{N}}\mathsf{CO}$ of $\mathcal{P}_{\mathbb{N}}$-coalgebras.

Theorem 5.1 *The functors* $R_{-} \colon \mathsf{CMVO} \to \mathcal{P}_{\mathbb{N}}\mathsf{CO}$ *and* $-_{*} \colon \mathcal{P}_{\mathbb{N}}\mathsf{CO} \to \mathsf{CMVO}$ *define a dual equivalence, where the associated natural isomorphisms* $e \colon I_{\mathsf{CMVO}} \to -_{*} \circ R_{-}$ *and* $\epsilon \colon I_{\mathcal{P}_{\mathbb{N}}\mathsf{CO}} \to R_{-} \circ -_{*}$ *are given by*

$$e_{\mathbf{A}} \colon x \mapsto e_{\mathbf{A}}(x) \colon a \mapsto i_a(x \wedge a),$$
$$\epsilon_R \colon x \mapsto \chi_{\{x\}},$$

for every $\mathbf{A} \in \mathsf{CMVO}$ *and* $R \in \mathcal{P}_{\mathbb{N}}\mathsf{CO}$*.*

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Plausibility and conditional beliefs in paraconsistent modal logic

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Abstract

We discuss a paraconsistent modal logic with modalities related to a plausibility preorder on the set of states. One of the modalities expresses support by all states at least as plausible as the present state, the other one is a form of a conditional belief. Our application of the formalism is an epistemic one and it involves the notion of support by the most plausible accessible states. The main technical result obtained so far is a completeness proof for a logic defined by the class of models satisfying a particular closure condition. An alternative, and potentially more general, approach to defining 'support by the most plausible accessible states'—one based on paraconsistent hybrid logic—is also discussed.

Keywords: Epistemic logic, Hybrid logic, Paraconsistent logic, Plausibility orders.

1 Introduction

Information available to cognitive agents is often inconsistent. Sometimes the inconsistency results from an error (in reasoning or observation) and sometimes it is a consequence of adopting information supplied by mutually inconsistent sources; it even happens that the trusted sources themselves provide inconsistent information to the agent.

Epistemic logics based on paraconsistent logic (see, e.g. [5]) are designed to formalize situations where the information available to an agent is inconsistent and yet non-trivial (i.e. the information, albeit inconsistent, does not support every possible conclusion). However, the existing systems do not represent the subtle distinctions between 'kinds of inconsistency' pointed out above.

In [7] an epistemic interpretation of the Belnapian modal logic BK (see [6]) was put forward and it was shown that the formalism can represent the inconsistency distinctions. The key to this is an interpretation of $\Diamond\phi$ as "ϕ is supported by a source for the given state"; one can then represent *justified* *support* of ϕ as $\phi \wedge \Diamond\phi$. Now a contradiction $\phi \wedge -\phi$ is 'unjustified' in a state if it supports the contradiction and it supports neither $\Diamond(\phi \wedge -\phi)$ nor $\Diamond\phi \wedge \Diamond-\phi$; the contradiction is justified by an inconsistent source for a state if the state supports $\Diamond(\phi \wedge -\phi)$ in addition to the contradiction; and the contradiction is justified by mutually inconsistent sources if the state supports $\Diamond\phi \wedge \Diamond-\phi$.

Especially in the latter case, when one source supports ϕ and another supports $-\phi$ the question of *relative reliability* of sources becomes relevant. Using the notion of reliability, we define our key modality "justification by the *best* sources" and a number of related notions.

In this short presentation we outline an extension of the framework presented in [7] with a *plausibility preorder* indicating relative reliability of sources and we introduce related modalities. In particular, $[\preceq]\phi$ expresses that ϕ is supported by every state that is at least as plausible as the present one; and $[\phi]\psi$ says that ψ is supported by each maximally plausible accessible ϕ-supporting state (a conditional belief operator). Hence, $[\top]\phi$, where $\top$ is a truth constant, says that ϕ is supported by each maximally plausible accessible state—formulas of this kind are especially important in our source-epistemic interpretation of paraconsistent modal logic. Our main technical result obtained so far is a completeness proof for a paraconsistent epistemic plausibility logic with conditional belief based on a special class of models. It is the focus of ongoing work to extend the result to paraconsistent plausibility logics defined using broader classes of models. Our approach here is to use a paraconsistent version of hybrid logic to define the modality corresponding to $[\top]$ directly.

2 A paraconsistent modal logic in a nutshell

We build on an epistemic interpretation, discussed in [7], of the paraconsistent modal logic BK introduced in [6] (we refer the reader to this paper for details about BK). The basic paraconsistent modal language $\mathcal{L}$ contains the usual connectives $\{\bot, -, \wedge, \vee, \rightarrow, \Box, \Diamond\}$. A BK-model is a Kripke frame $\langle S, R\rangle$ extended with a pair of valuations V^+, V^- such that $V^+(p)$ is the set of states that support truth of p and $V^-(p)$ is the set of states that support falsity of p. Some states may give no information about p at all ($V^+(p) \cup V^-(p) \neq S$), while some may give contradictory information about p ($V^+(p) \cap V^-(p) \neq \emptyset$). The valuations are extended to a *truth relation* $\vDash^+$ and a *falsity relation* $\vDash^-$ in an expected manner (see [6]):

$x \vDash^+ p$ iff $x \in V^+(p)$	$x \vDash^- p$ iff $x \in V^-(p)$
$x \vDash^+ -\phi$ iff $x \vDash^- \phi$	$x \vDash^- -\phi$ iff $x \vDash^+ \phi$
$x \vDash^+ \phi \wedge \psi$ iff $x \vDash^+ \phi$ and $x \vDash^+ \psi$	$x \vDash^- \phi \wedge \psi$ iff $x \vDash^- \phi$ or $x \vDash^- \psi$
$x \vDash^+ \phi \rightarrow \psi$ iff $x \nvDash^+ \phi$ or $x \vDash^+ \psi$	$x \vDash^- \phi \rightarrow \psi$ iff $x \vDash^+ \phi$ and $x \vDash^- \psi$

$x \vDash^+ \Box\phi$ iff for all y, if Rxy, then $y \vDash^+ \phi$
$x \vDash^- \Box\phi$ iff there is y such that Rxy and $y \vDash^- \phi$
$x \vDash^+ \Diamond\phi$ iff there is y such that Rxy and $y \vDash^+ \phi$
$x \vDash^- \Diamond\phi$ iff for all y, if Rxy, then $y \vDash^- \phi$

ϕ is valid in a model iff $x \vDash^+ \phi$ for all states x in the model. Validity in a frame, a class of models and a class of frames are defined as usual. BK is the class of formulas valid in all frames.

In [7], states x were seen, informally, as bodies of information that *support* formulas ($x \vDash^+ \phi$ means that x supports ϕ). These bodies of information may be incomplete and inconsistent so, for example, x may support both ϕ and $-\phi$, the negation of ϕ. The accessibility relation R is seen as a *source* relation (cf. also [2]). Informally, Rxy means that y is a source (of information) for x; if the information provided by x needs to be *justified*, one looks at the sources of x to see whether any one of them supports the information in question. Therefore, the formula $\phi \wedge \Diamond\phi$ says that ϕ is *justified*—ϕ is supported by the present state and also by one of the sources of the present state.

This interpretation allows us to express distinctions between different kinds of contradictory information (contradictions unjustified by sources vs. contradictory information provided by distinct sources vs. by a single source).

We may read states in paraconsistent models as bodies of information available to agents, but this is not our official interpretation (we adopt a more abstract reading that does not involve agents).

3 Adding plausibility

In situations where distinct sources provide mutually inconsistent information it is natural to ask about the *relative plausibility* (or reliability) of the sources. This motivates a combination of BK with modal logics containing operators based on plausibility orderings on states. (These go back to Lewis' logic of counterfactuals, for more recent approach see [1].)

A (paraconsistent) plausibility model is $\mathfrak{M} = \langle S, \preceq, R, V^+, V^- \rangle$ such that $\langle S, \preceq \rangle$ is a preordered set and $\langle S, R, V^+, V^- \rangle$ is a BK model. A model is *simple* if it satisfies the 'preorder closure condition'

$$Rxy \text{ and } y \preceq z \implies Rxz \tag{1}$$

(Every body of information at least as reliable as a source is also a source). We denote as $Max_{\preceq}X$ the set of maximal members of X (under $\preceq$). We extend the language $\mathcal{L}$ with modalities $\{[\preceq], \langle\preceq\rangle, [\phi], \langle\phi\rangle\}$ (for all formulas ϕ) with the following truth/falsity conditions:

$x \vDash^+ [\phi]\psi$ iff for all $y \in Max_{\preceq}(R(x))$, $y \vDash^+ \phi$ implies $y \vDash^+ \psi$

$x \vDash^- [\phi]\psi$ iff there is some $y \in Max_{\preceq}(R(x))$ such that $y \vDash^+ \phi$ and $y \vDash^- \psi$

$x \vDash^+ [\preceq]\phi$ iff for all $y \succeq x$, $y \vDash^+ \phi$

$x \vDash^- [\preceq]\phi$ iff for some $y \succeq x$, $y \vDash^- \phi$

The conditions for the diamond are as expected. Validity is defined as before. $BK_s^{\preceq}$ is the set of formulas valid in all simple (plausibility) frames. Note that $Max_{\preceq}(R(x) \cap |\phi|)$ might be empty, in that case $x \vDash^+ [\phi]\psi$ for every ψ.

Theorem 3.1 *$\phi \in BK_s^{\preceq}$ iff ϕ is a theorem of the axiom system that is obtained by adding the following to the axiom system for BK (see [6]):*

- *S4 axioms for* $[\preceq]$

- *the following axioms:*
  $\Box\phi \to [\psi]\phi$ $\quad$ $[\phi]\phi$
  $\Box\phi \to \Box[\preceq]\phi$ $\quad$ $[\phi]\psi \to \Box(\phi \to [\preceq](\phi \to \psi))$
  $[\phi](\langle\preceq\rangle(\phi \wedge \psi) \to \psi)$
- *the rule* $\dfrac{\phi_1 \wedge \ldots \wedge \phi_n \to \psi}{[\chi]\phi_1 \wedge \ldots \wedge [\chi]\phi_n \to [\chi]\psi}$

In the extended setting, we can express 'ϕ is supported by all of the most reliable (plausible) sources' as $[\top]\phi$. Therefore, we may express ($\neg$ is classical negation defined as $\phi \to \bot$, see [6]):

- $\phi \wedge \Diamond(\phi \wedge [\preceq]\phi)$, '$\phi$ is supported and justified by a source; moreover, ϕ is supported by all states that are at least as reliable as that source'
- $\phi \wedge \Diamond(\phi \wedge \neg\langle\preceq\rangle{-\phi})$, '$\phi$ is supported and justified by a source; moreover, ϕ is not rejected by any source that is at least as reliable as that source'
- $\phi \wedge [\top]\phi$, 'ϕ is supported and justified by all of the most reliable sources'

If we want to axiomatize our notion of most plausible sources, we have to increase radically the expressivity of our background logic.

4 Hybrid framework

Hybrid languages extend standard modal logic with a second sort of propositional symbols with restricted interpretation - nominals. A nominal is true at exactly one state of a Kripke model, we can see it as a name of the state it is true at. For example the (two sorted) formula $i \wedge p$ says that p is true at the state i. Nominals are accompanied with the @ operator, which allows us to point to a particular state: $@_i\phi$ is true iff ϕ is true at the state named i. The resulting language is strictly more expressive than that of standard modal logic, it can express properties not definable in the standard modal language (e.g. asymmetry: $@_i\neg\Box\Box_i$).

In order to get sufficient expressive power we need to use an extension of the basic hybrid language with the *binder* $\downarrow$, which allows us to assign a "label" to the "current" state (state of evaluation). We also assume our language contains an infinite set of *state variables.*

With this equipment at hand the definition of the most plausible sources just translates the corresponding first order formula into the hybrid language:

$$\Box^{max}\phi \equiv_{def} \downarrow w(\Diamond \downarrow x(\phi \wedge @_w\Box \downarrow z(\ @_x[\preceq]z \to @_z x)))$$

Our presentation of the hybrid framework builds on the article [4]. We extend the language $\mathcal{L}^{\preceq}$ with the additional symbols of the hybrid language:

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid [\preceq]\phi \mid\ i \mid s \mid @_i \mid @_s \mid\ \downarrow_s \phi$$

Hybrid plausibility models are plausibility models such that valuation is extended to nominals in a way that $V^+(i) = V^+(j)$ only if $i = j$ and each $V^+(i)$ is a singleton. Moreover, we need a function g from the set of state variables to the set of states S. We extend the satisfaction conditions with

(i) $g, x \models^+ i$ iff $\{x\} = V^+(i)$

(ii) $g, x \models^+ @_i\phi$ iff $y \models^+ \phi$, where $\{y\} = V^+(i)$

(iii) $g, x \models^+ @_s\phi$ iff $g, y \models^+ \phi$, where $y = g(s)$

(iv) $g, x \models^+ \downarrow_s \phi$ iff $g', x \models^+ \phi$, where $x = g'(s)$ and g' is an assignment which possibly differs from g only with respect to s

Falsity conditions are defined in an obvious way. Our axiomatization of the hybrid epistemic logic of "most plausible sources" consists of axioms and rules for

(CL)	propositional logic	($\Box$)	normal modal logic $\Box$
($\preceq$)	S4 modality $[\preceq]$	(H)	hybrid logic $H(@\downarrow)$

plus the axioms regulating interaction between @ and our second modality

- $\Diamond @_i\phi \to @_i\phi$
- $\dfrac{@_i\langle\preceq\rangle j \to @_j\phi}{@_i[\preceq]\phi}$, where $j \neq i$ is a nominal that does not occur in ϕ

Blackburn and ten Cate prove that the axiomatization of $H(@\downarrow)$ ((CL), ($\Box$) (H) in our notation) is sound and complete with respect to the class of hybrid frames, and moreover completeness is guaranteed for arbitrary extensions with pure axioms, i.e. those where the only atomic formulas are nominals ([3], Theorem 5). This is exactly the case of our additional axioms (characterizing partial orders – reflexivity, transitivity and symmetry). The canonical model construction in the completeness proof can be straightforwardly adopted to the four-valued setting and we can prove the completeness of our axiomatization with respect to hybrid plausibility models.

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Finite Model Properties for the One-Variable Fragment of First-Order Gödel logic

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1 Introduction

It is well-known that the one-variable fragments of classical and intuitionistic first-order logic, understood also as notational variants of the modal logic $\mathsf{S5}$ and Bull and Prior's intuitionistic modal logic MIPC, respectively, are decidable and have the finite model property. Recently, it was proved that the one-variable fragment of first-order Gödel logic, a notational variant of the many-valued modal logic $\mathsf{S5}^{\mathsf{c}}(\mathsf{G})$ studied in [7,8,3], is also decidable (indeed co-NP complete), despite lacking the finite model property with respect to its standard semantics [2]. The key step in the decidability proof is the definition of an alternative semantics for $\mathsf{S5}^{\mathsf{c}}(\mathsf{G})$, restricting the values of box and diamond formulas, that does admit the finite model property.

In this work we show that this alternative semantics, defined on a rather ad hoc basis in [2], emerges naturally from a representation of monadic Heyting algebras (studied in some depth in [1]) as pairs consisting of a Heyting algebra and a relatively complete subalgebra. As a consequence, we obtain an algebraic finite model property for the variety of monadic Heyting algebras corresponding to $\mathsf{S5}^{\mathsf{c}}(\mathsf{G})$ and shorter, more elegant, proofs of the decidability and co-NP-completeness results obtained in [2].

2 The One-Variable Fragment

Propositional Gödel logic may be defined semantically for the usual language of intuitionistic logic over a countably infinite set Var of variables as the logic of the algebra $\mathbf{G} = \langle [0,1], \min, \max, \to_{\mathbf{G}}, 0, 1\rangle$ with designated value 1, where

$$a \to_{\mathbf{G}} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

In addition to being an important many-valued logic, Gödel logic has also been studied extensively as an intermediate logic, axiomatized by extending intuitionistic logic with the prelinearity axiom $(\varphi \to \psi) \lor (\psi \to \varphi)$ [5].

First-order Gödel logic is defined semantically based on models defined over the algebra $\mathbf{G}$ and axiomatized (see [9]) by extending first-order intuitionistic logic with both the prelinearity axiom and the constant domain axiom

$$(\forall x)(\varphi \vee \psi) \to (\varphi \vee (\forall x)\psi), \quad x \text{ not free in } \varphi.$$

The logic obtained by dropping this second axiom is known as first-order Corsi logic and is complete with respect to Kripke models based on linearly ordered frames with nested domains (see [4]).

We focus in this work on the *one-variable fragment* of first-order Gödel logic, a notational variant of the many-valued modal logic $\mathsf{S5^c(G)}$ obtained by replacing each predicate $P(x)$ with a propositional variable p, and quantifiers $\forall$ and $\exists$ with $\Box$ and $\Diamond$, respectively. More precisely, a (universal) $\mathsf{S5^c(G)}$-*model* [1] is a pair $\mathcal{M} = \langle W, V\rangle$ consisting of a non-empty set of worlds W and a valuation function $V \colon \mathsf{Var} \times W \to [0,1]$. The valuation V is extended to all formulas by $V(0,x) = 0$, $V(1,x) = 1$, and

$$\begin{aligned}
V(\varphi \wedge \psi, x) &= \min(V(\varphi,x), V(\psi,x))\\
V(\varphi \vee \psi, x) &= \max(V(\varphi,x), V(\psi,x))\\
V(\varphi \to \psi, x) &= V(\varphi,x) \to_{\mathbf{G}} V(\psi,x)\\
V(\Box\varphi, x) &= \bigwedge\{V(\varphi,y) \mid y \in W\}\\
V(\Diamond\varphi, x) &= \bigvee\{V(\varphi,y) \mid y \in W\}.
\end{aligned}$$

A formula φ is called *valid* in $\mathcal{M}$ if $V(\varphi,x) = 1$ for all $x \in W$, and we say that φ is $\mathsf{S5^c(G)}$-*valid* if it is valid in all $\mathsf{S5^c(G)}$-models.

Caicedo and Rodríguez (following a previous result of Hájek [8]) proved in [3] that $\mathsf{S5^c(G)}$ can be axiomatized as an extension of the modal intuitionistic logic MIPC with the prelinearity axiom and modal analogue of the constant domain axiom $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$. Unlike MIPC and $\mathsf{S5}$, however, $\mathsf{S5^c(G)}$ does not admit the finite model property with respect to its Kripke model semantics. For example, the formula $\Diamond(p \to \Box p)$ is valid in all finite $\mathsf{S5^c(G)}$-models, but not the infinite $\mathsf{S5^c(G)}$-model $\langle \mathbb{N}, V\rangle$ where $V(p,n) = \frac{1}{n+1}$; just observe that

$$\begin{aligned}
V(\Diamond(p \to \Box p), 0) &= \bigvee_{n\in\mathbb{N}} V(p \to \Box p, n)\\
&= \bigvee_{n\in\mathbb{N}} \big(V(p,n) \to_{\mathsf{G}} \bigwedge_{m\in\mathbb{N}} V(p,m)\big)\\
&= \bigvee_{n\in\mathbb{N}} \big(\tfrac{1}{n+1} \to_{\mathsf{G}} \bigwedge_{m\in\mathbb{N}} \tfrac{1}{m+1}\big)\\
&= \bigvee_{n\in\mathbb{N}} \big(\tfrac{1}{n+1} \to_{\mathsf{G}} 0\big)\\
&= 0.
\end{aligned}$$

[1] The superscript here stands for "crisp" and reflects the fact that Kripke models for Gödel modal logics may also be defined with a many-valued relation, that is, a function $R\colon W^2 \to [0,1]$ (see [2] for further details).

Nevertheless, as will be explained below, $\mathsf{S5^c(G)}$ does admit an *algebraic* finite model property.

3 An Algebraic Perspective

An algebra $\langle H, \wedge, \vee, \to, 0, 1, \Box, \Diamond\rangle$ (also shortened to $\langle \mathbf{H}, \Box, \Diamond\rangle$) is called a *monadic Heyting algebra* if $\mathbf{H} = \langle H, \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra and $\Box, \Diamond$ are unary operators on H satisfying for all $a, b \in H$,

(1a) $\Box a \le a$	(1b) $a \le \Diamond a$
(2a) $\Box(a \wedge b) = \Box a \wedge \Box b$	(2b) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
(3a) $\Box 1 = 1$	(3b) $0 = \Diamond 0$
(4a) $\Box\Diamond a = \Diamond a$	(4b) $\Diamond\Box a = \Box a$
(5a) $\Diamond(\Diamond a \wedge b) = \Diamond a \wedge \Diamond b$.	

A monadic Heyting algebra satisfying the prelinearity law $(a \to b) \vee (b \to a) = 1$ and constant domain law $\Box(\Box a \vee b) = \Box a \vee \Box b$ is called a *monadic Gödel algebra.*

It is easy to prove that the logics MIPC and $\mathsf{S5^c(G)}$ are sound and complete with respect to the classes of monadic Heyting algebras and monadic Gödel algebras, respectively; indeed, the lattices of axiomatic extensions of MIPC and varieties of monadic Heyting algebras are dual (see [1]). Such algebras also admit a useful alternative representation. Given any monadic Heyting algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$, the set $H_0 = \{\Box a \mid a \in H\} = \{\Diamond a \mid a \in H\}$ forms a subuniverse of $\mathbf{H}$ such that for any $a \in H$,

$$\Box a = \bigvee\{b \in H_0 \mid b \le a\} \quad \text{and} \quad \Diamond a = \bigwedge\{b \in H_0 \mid b \ge a\}.$$

Moreover, let us call any subuniverse H_0 of a Heyting algebra $\mathbf{H}$ where all such suprema and infima exist in H_0 *relatively complete.* If $\Box$ and $\Diamond$ are defined as described above for such an H_0, then $\langle \mathbf{H}, \Box, \Diamond\rangle$ is a monadic Heyting algebra. Hence we obtain the following equivalence.

Theorem 3.1 ([1]) *There exists a one-to-one correspondence between*

(i) *monadic Heyting algebras* $\langle \mathbf{H}, \Box, \Diamond\rangle$*;*

(ii) *the pairs* $\langle \mathbf{H}, \mathbf{H}_0\rangle$ *of Heyting algebras where* $\mathbf{H}_0$ *is a relatively complete subalgebra of* $\mathbf{H}$.

Moreover, this correspondence can be extended to a categorical equivalence.

We now apply these ideas to investigate monadic Gödel algebras. Let us call a monadic Gödel algebra *standard* if it is of the form $\langle \mathbf{G}^W, \Box, \Diamond\rangle$, where W is any non-empty set, $\mathbf{G}^W$ is the Heyting algebra with universe $[0,1]^W$ and operations defined pointwise, and for each $f \in [0,1]^W$ and $x \in W$,

$$\Box(f)(x) = \bigwedge\{f(y) \mid y \in W\}$$
$$\Diamond(f)(x) = \bigvee\{f(y) \mid y \in W\}.$$

Using the completeness result of Caicedo and Rodríguez [3] mentioned in the previous section, a formula φ is $\mathsf{S5^c(G)}$-valid if and only if $\varphi = 1$ is valid in all standard monadic Gödel algebras. Equivalently, the variety of monadic Gödel algebras is generated (as a variety) by its standard members.

Observe now that for any standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond \rangle$, the subuniverse $\{\Box(f) \mid f \in [0,1]^W\}$ consists of all constant functions for $r \in [0,1]$,

$$f_r \colon W \to [0,1];\ x \mapsto r.$$

In order to establish a finite model property (which fails for standard monadic Gödel algebras), we consider the class of monadic Gödel algebras obtained by allowing also proper subsets of these constant functions. More precisely, for any complete sublattice T of $[0,1]$ containing $\{0,1\}$, the set $\{f_r \mid r \in T\}$ is a relatively complete subuniverse of $\mathbf{G}^W$ and yields a monadic Gödel algebra with modal operators

$$\begin{aligned} \Box(f)(x) &= \bigvee\{r \in T \mid r \leq \bigwedge\{f(y) \mid y \in W\}\} \\ \Diamond(f)(x) &= \bigwedge\{r \in T \mid r \geq \bigvee\{f(y) \mid y \in W\}\}. \end{aligned}$$

These definitions correspond exactly to the alternative semantics used in [2] to prove decidability and complexity results for $\mathsf{S5^c(G)}$. Moreover, we obtain simpler proofs of these results (avoiding a rather complicated "squeezing" of truth values argument) by establishing a finite model property with respect to this class of monadic Gödel algebras and noting that it must also (since standard monadic Gödel algebras form a subclass) generate (as a variety) the whole variety of monadic Gödel algebras.

We make use of the following lemma, established by fixing T as the finite set of values taken by subformulas of a formula failing in some world of a standard monadic Gödel algebra, then choosing finitely many worlds that "witness" these values for the box and diamond subformulas.

Lemma 3.2 *If $\varphi \approx 1$ is not valid in a standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond \rangle$, then it is not valid in a monadic Gödel algebra corresponding to $\langle \mathbf{G}^{W'}, \{f_r \mid r \in T\} \rangle$ for some finite $W' \subseteq W$ and finite $T \subseteq [0,1]$.*

Note that this lemma expresses a finite model property with respect to the number of worlds, but the algebras themselves still have an infinite universe of the form $[0,1]^W$. A genuinely algebraic finite model property is obtained by observing that only finitely many truth values are required to provide a countermodel for a formula; an analysis of the number of truth values required then also yields an upper complexity bound for checking validity in $\mathsf{S5^c(G)}$.

Theorem 3.3 *The variety of monadic Gödel algebras has the finite model property, and checking validity in $\mathsf{S5^c(G)}$ is decidable, indeed co-NP-complete.*

The approach described here can also be used to prove algebraic finite model property, decidability, and co-NP-completeness results for the many-valued modal logic $\mathsf{S5(G)}$, defined via models with a many-valued accessibility relation and axiomatized as MIPC extended with the prelinearity axiom [3]. It can also be proved that this logic corresponds exactly to the one-variable fragment of first-order Corsi logic, axiomatized in [4] by adding the prelinearity axiom to first-order intuitionistic logic. Note, however, that the algebraic finite model property and decidability in this case follow already as a consequence of general results for varieties of monadic Heyting algebras provided in [1].

Let us note finally that there exist methods for translating intermediate modal logics into classical bimodal logics; it is then possible to import finite model, decidability, and complexity results from the classical domain (see [6]). Such methods do not directly apply here for two reasons. Firstly, they focus on the box fragments of these logics, and, secondly, they rely on a presentation of intermediate modal logics that uses two relations, one for the intuitionistic connectives and one for the modality. Nevertheless, it may be useful to adapt the approach described here to the classical bimodal logic setting, and we leave this as an interesting topic for future work.

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Existentially valid formulas corresponding to some normal modal logics

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Abstract

Validity of a modal formula on a Kripke frame is defined as global truth under any valuation of propositional variables. Analogously, we can consider so-called existential validity of formulas, i.e. satisfiability under any valuation. We outline relationships between sets of existentially valid formulas corresponding to several well-known modal logics.

Keywords: modal logic, Kripke semantics, existential validity.

1 Introduction

We say that a modal formula is valid on a frame if it is true at any world under any valuation of propositional variables, i.e. if it is globally true under any valuation. Humberstone [4] studies an analogous notion of existential validity: we say that a formula is $\exists$-*valid* if it is true at *some* world under any valuation (i.e. satisfiable under any valuation). In [4] it is shown that the set of all existentially valid formulas (so-called $\exists$-logic) can be axiomatized.

Clearly, $\exists$-validity corresponds to the usual validity of formulas belonging to the existential fragment of modal logic enriched with the universal modality, studied e.g. in [3]. An analogue of Goldblatt-Thomason theorem for $\exists$-validity is given in [6]. In the main section we compare the $\exists$-logics corresponding to some well-known systems of modal logic.

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2 Comparison of some $\exists$-logics

We will use the following traditional notation for modal formulas which define some important properties of frames:

Notation	Formula A	Frame condition (A)
T	$\Box p \to p$	xRx for all x (*reflexivity*)
B	$p \to \Box\Diamond p$	if xRy then yRx (*symmetry*)
4	$\Box p \to \Box\Box p$	if xRy and yRz then xRz (*transitivity*)
5	$\Diamond p \to \Box\Diamond p$	if xRy and xRz then yRz (R is *Euclidean*)
D	$\Box p \to \Diamond p$	for each x there is y s.t. xRy (*seriality*)

It is well-known that by extending the basic system of normal modal logic **K** with some of the above formulas we obtain systems sound and complete with respect to frames satisfying the respective frame conditions. These frames we call **L**-*frames*, where **L** is such a system. Some combinations coincidentally generate the same systems, so we consider the fifteen distinct systems. We suggest the cube-shaped diagram in [2] (section 8) as a reference. We denote the set of all formulas $\exists$-valid on all **L**-frames by $\mathbf{L}^{\exists}$. Note that this set is defined semantically. The definition implies that $\mathbf{L} \subseteq \mathbf{L}^{\exists}$, for all **L**. The set $\mathbf{K}^{\exists}$ was axiomatized in [4] where it was denoted as $\dot{\mathbf{K}}$.[2]

The following simple observation provides the basic information regarding relationships of $\exists$-validity of distinct logics.

Fact 2.1 *If* $\mathbf{L}_1 \subseteq \mathbf{L}_2$, *then* $\mathbf{L_1^{\exists}} \subseteq \mathbf{L_2^{\exists}}$.

We will need the following Lemma.

Lemma 2.2 ([5]; [7]; [1]) *Let $\mathfrak{M}$ be an model and φ a formula. If $\mathfrak{M} \Vdash \varphi$, and $\mathfrak{M}$ has some of the properties (T), (B), (4), (5) and (D), then there is a finite model $\mathfrak{M}'$ having the same properties and such that $\mathfrak{M}' \Vdash \varphi$.*[3]

The idea behind the following lemma was used in [8], in the proof that **S4**-consistent formulas are exactly the **S5**-consistent formulas.

Lemma 2.3 *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two of the fifteen logics obtained by extending* **K** *with some of the formulas T, B, 4, 5 and D. Assume that every finite $\mathbf{L}_1$-frame has a generated $\mathbf{L}_2$-subframe. Then $\mathbf{L_2^{\exists}} \subseteq \mathbf{L_1^{\exists}}$.*

Proof. Let $\varphi \in \mathbf{L_2^{\exists}}$. Let $\mathfrak{N}$ be an $\mathbf{L}_1$-model. Assume $\mathfrak{N} \Vdash \neg\varphi$. Then Lemma 2.2 implies there is a finite $\mathbf{L}_1$-model $\mathfrak{M}$ with $\mathfrak{M} \Vdash \neg\varphi$. Our assumption implies that $\mathfrak{M}$ has a generated $\mathbf{L}_2$-submodel $\mathfrak{M}'$.

Since the global truth of modal formulas is preserved under taking generated submodels, we have that $\mathfrak{M}' \Vdash \neg\varphi$, contradicting the assumption that $\varphi \in \mathbf{L_2^{\exists}}$.$\Box$

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[2] A reviewer of the first version of this paper suggested changing the notation to increase readability.

[3] The property stated in this lemma is what is usually known as the *global* finite model property: we require that for each φ and **L**-model $\mathfrak{M}$ with $\mathfrak{M} \Vdash \varphi$ (not $\mathfrak{M}, w \Vdash \varphi$ for some w) there is a finite **L**-model $\mathfrak{M}'$ with $\mathfrak{M}' \Vdash \varphi$ (not only $\mathfrak{M}', w \Vdash \varphi$ for some w). The proofs of the finite model property and the global finite model property are exactly the same in the case of the fifteen logics considered.

We will now use the results above to analyze relationships between the fifteen logics we consider.

Proposition 2.4

$$\mathbf{K4}^{\exists} = \mathbf{K5}^{\exists} = \mathbf{K45}^{\exists} = \mathbf{K4B}^{\exists}.$$

Proof. Fact 2.1 implies: $\mathbf{K4}^{\exists}, \mathbf{K5}^{\exists} \subseteq \mathbf{K45}^{\exists} \subseteq \mathbf{K4B}^{\exists}$.

Thus it suffices to prove $\mathbf{K4B}^{\exists} \subseteq \mathbf{K4}^{\exists} \subseteq \mathbf{K5}^{\exists}$. We will first prove that $\mathbf{K4B}^{\exists} \subseteq \mathbf{K4}^{\exists}$.

In order to use Lemma 2.3, we should be able to find, in each finite **K4**-frame, a generated **K4B**-subframe.

Let $\mathfrak{F} = (W, R)$ be a finite **K4**-frame. Let $w \in W$ be a maximal world, that is, for all x, if wRx, then xRw. Finiteness and transitivity guarantee the existence of such w. Let $\mathfrak{F}' = (W', R')$ be a subframe generated by $\{w\}$ (since R is transitive, $W' = \{w\} \cup R(w)$, where $R(w) = \{x : wRx\}$). Let us prove that $\mathfrak{F}'$ satisfies (B) (symmetry). Assume $xR'y$ (thus, $w = xR'y$ or $wRxRy$). The maximality of w, and wRy, imply yRw, thus $yR'w$. The transitivity of R' implies $yR'x$; thus $\mathfrak{F}'$ is indeed a **K4B**-frame.

Now, let $\mathfrak{F} = (W, R)$ be a finite **K5**-frame. Let $w \in W$ be arbitrary. Let $\mathfrak{F}' = (W', R')$ be the subframe generated by w if $R(w) = \emptyset$, otherwise the subframe generated by $R(w)$. Let us prove that $\mathfrak{F}'$ satisfies (4) (transitivity). If $R(w) = \emptyset$, (4) holds. Otherwise, assume $xR'yR'z$. Thus, wR^kxRyRz for some $k \geq 1$. Let t be the immediate predecessor of x in this chain, i.e. $tRxRyRz$. Now, tRx (and tRx) and (5) imply xRx. From this, xRy and (5) we have yRx. Finally, yRx and yRz imply xRz. Thus, $\mathfrak{F}'$ satisfies (4), and so $\mathfrak{F}'$ is a **K4**-frame. □

Proposition 2.5

$$\mathbf{D4}^{\exists} = \mathbf{D5}^{\exists} = \mathbf{S4}^{\exists} = \mathbf{S5}^{\exists} = \mathbf{D45}^{\exists}.$$

Proof. We know (see [2]) that $\mathbf{D4}, \mathbf{D5} \subseteq \mathbf{D45} \subseteq \mathbf{S5} = \mathbf{D4B}$. In [8] it is proved that $\mathbf{S4}^{\exists} = \mathbf{S5}^{\exists}$. (The proof is analogous to the proof of Proposition 2.4). So it suffices to show that $\mathbf{D4B}^{\exists} \subseteq \mathbf{D4}^{\exists} \subseteq \mathbf{D5}^{\exists}$. But the proof of this is analogous to the proof of a similar claim in Proposition 2.4. Simply assume that $\mathfrak{F}$ satisfies (D) (seriality), and note that any generated subframe of a serial frame must also be serial. □

Having these equalities, it remains to verify relationships between the following logics:

$$\mathbf{K}, \mathbf{KB}, \mathbf{K4}, \mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{TB}, \mathbf{D4}.$$

For $F \in \{4, T, B, D\}$ denote $F^+ := F \wedge \Box F \wedge \Box\Box F$. It is easy to see that if $\mathfrak{F}$ has (F), then $\mathfrak{F} \vDash F^+$.

Let the frames a, b and c be defined as depicted on Fig. 2.

Lemma 2.6 *We have:*[4] *(i)* $\mathbf{D}^{\exists} \not\subseteq \mathbf{K4}^{\exists}$, *(ii)* $\mathbf{T}^{\exists} \not\subseteq \mathbf{DB}^{\exists}$, *(iii)* $\mathbf{KB}^{\exists} \not\subseteq \mathbf{T}^{\exists}$, *and (iv)* $\mathbf{K4}^{\exists} \not\subseteq \mathbf{TB}^{\exists}$.

[4] The first version of this article stated only the first claim and its consequences. After

Proof.

(i) Obviously, any one-point irreflexive frame is a $\mathbf{K4}$-frame, and $\Diamond\top$ is not satisfiable in it. Since $\Diamond\top \in \mathbf{D}$, we have $\Diamond\top \in \mathbf{D}^{\exists} \setminus \mathbf{K4}^{\exists}$.

(ii) Note that (a) is a $\mathbf{DB}$-frame. Let $\mathfrak{M}$ be a model based on this frame with $V(p) := \{x\}$. It is easy to see that T^+ is not satisfiable in $\mathfrak{M}$. However, $T^+ \in \mathbf{T}$.

(iii) Note that (b) is a $\mathbf{T}$-frame. Let $\mathfrak{M}$ be a model based on this frame with $V(p) := \{x, y\}$. It is easy to see that B^+ is not satisfiable in $\mathfrak{M}$.

(iv) Note that (c) is a $\mathbf{TB}$-frame. Let $\mathfrak{M}$ be a model based on this frame with $V(p) := \{x, y\}$. It is easy to see that 4^+ is not satisfiable in $\mathfrak{M}$.

$\square$

Corollary 2.7 *We have:*

(i) *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{K4}\}$ *and* $\mathbf{L_2} \in \{\mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{TB}, \mathbf{D4}\}$*, then* $\mathbf{L_2^{\exists}} \not\subseteq \mathbf{L_1^{\exists}}$.

(ii) *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{D}, \mathbf{DB}\}$ *and* $\mathbf{L_2} \in \{\mathbf{K4}, \mathbf{T}, \mathbf{TB}, \mathbf{D4}\}$*, then* $\mathbf{L_2^{\exists}} \not\subseteq \mathbf{L_1^{\exists}}$.

(iii) *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{D}, \mathbf{T}\}$ *and* $\mathbf{L_2} \in \{\mathbf{KB}, \mathbf{K4}, \mathbf{DB}, \mathbf{TB}, \mathbf{D4}\}$*, then* $\mathbf{L_2^{\exists}} \not\subseteq \mathbf{L_1^{\exists}}$.

(iv) *If* $\mathbf{L_1} \in \{\mathbf{K}, \mathbf{KB}, \mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{TB}\}$ *and* $\mathbf{L_2} \in \{\mathbf{K4}, \mathbf{D4}\}$*, then* $\mathbf{L_2^{\exists}} \not\subseteq \mathbf{L_1^{\exists}}$.

This completes the picture of relationships between sets $\mathbf{L}^{\exists}$ for the traditional logics $\mathbf{L}$: the fifteen traditional logics collapsed into eight $\exists$-logics.

$$\begin{tikzcd}
\mathbf{T}^{\exists} \arrow[r] & \mathbf{TB}^{\exists} \arrow[r] & \mathbf{D4}^{\exists} \\
\mathbf{D}^{\exists} \arrow[u] \arrow[r] & \mathbf{DB}^{\exists} \arrow[u] & \\
\mathbf{K}^{\exists} \arrow[u] \arrow[r] & \mathbf{KB}^{\exists} \arrow[u] \arrow[r] & \mathbf{K4}^{\exists} \arrow[uu]
\end{tikzcd}$$

Fig. 1. Arrows represent proper subsets.

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submitting we proved the other three hold. One of the anonymous reviewers proposed the following alternative claims. Their purpose, as in our version, is to show that all the inclusions in Fig. 1 are proper. Consider the following formulas:

(ii) Denote $\mathsf{Ref}(q) = \Box q \to q$. Take the formula $A = \mathsf{Ref}(p) \wedge \mathsf{Ref}(\neg p)$. Then $A \in \mathbf{T} \subseteq \mathbf{T}^{\exists}$, but $A \notin \mathbf{DB}^{\exists}$, because A is *not* satisfiable in the $\mathbf{DB}$-model $\mathfrak{M}$ based on the frame (a) with the valuation $(p, \neg p)$, since $\mathfrak{M} \vDash \neg A$. Thus $\mathbf{T}^{\exists} \not\subseteq \mathbf{DB}^{\exists}$.

(iii) Denote $\mathsf{Sym}(q) = q \to \Box\Diamond q$. Take the formula $A' = \mathsf{Sym}(\alpha) \wedge \mathsf{Sym}(\beta) \wedge \mathsf{Sym}(\gamma)$, where $\alpha = p \wedge \Box p$, $\beta = p \wedge \Diamond\neg p$, $\gamma = \neg p$. Then $A' \in \mathbf{KB} \subseteq \mathbf{KB}^{\exists}$, but $A' \notin \mathbf{T}^{\exists}$, because A' is *not* satisfiable in the $\mathbf{T}$-model $\mathfrak{M}'$ based on the frame (b) with the valuation $(p, p, \neg p)$ since $\mathfrak{M}' \vDash \neg A'$. Thus $\mathbf{KB}^{\exists} \not\subseteq \mathbf{T}^{\exists}$.

(iv) Denote $\mathsf{Tr}(q) = \Box q \to \Box\Box q$. Take the formula $A'' = \mathsf{Tr}(p) \wedge \mathsf{Tr}(\neg p) \wedge \mathsf{Tr}(\Diamond p) \wedge \mathsf{Tr}(\Diamond\neg p)$. Then $A'' \in \mathbf{K4} \subseteq \mathbf{K4}^{\exists}$, but $A'' \notin \mathbf{TB}^{\exists}$, because A'' is *not* satisfiable in the $\mathbf{TB}$-model $\mathfrak{M}''$ based on a frame like (c) but with 4 points, with the valuation $(p, p, \neg p, \neg p)$, since $\mathfrak{M}'' \vDash \neg A''$. Thus $\mathbf{K4}^{\exists} \not\subseteq \mathbf{TB}^{\exists}$

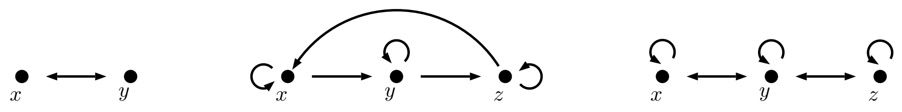

Fig. 2. Frame (a) satisfies (B) and (D), but not (T), (4) and (5). Frame (b) satisfies (D) and (T), but not (B), (4) and (5). Frame (c) satisfies (B), (D) and (T), but not (4) and (5).

3 Further work

Zolin [9] posed the question of axiomatizability of ∃-logics over classes of frames other then the class of all frames, for which this question is answered in [4]. As a step towards this, we have addressed the question of distinction between the ∃-logics corresponding to the well-known modal systems, leaving further research for the future work.

Furthermore, there are computational aspects that we can consider, for example the question of complexity of deciding if a given formula is ∃-valid on all frames with a certain property (or dually, if a given formula is true in all worlds of some model with this property – this is the so-called global satisfiability), or an analogous question w.r.t. a restricted, for example finite-variable, fragment.

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Failure of Interpolation in Stit Logic

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Abstract

We show that Craig Interpolation Property fails for basic stit logic with at least 4 agents in rather strong sense. Moreover, Strong Craig Interpolation Property fails for the basic stit logic whenever more than 1 agent is present.

Keywords: Stit logic, Interpolation

The results to be reported in the talk concern the language of propositional logic equipped with *Chellas stit* (cstit) operators for a finite set of individual agents plus the historical necessity modality. The full version of the proofs will be published as [3].

More precisely, for a given a finite agent community Ag and a set of propositional variables V, we consider the language $\mathcal{L}^{Ag}_V$ of (Ag, V)-stit formulas as given by the following BNF:

$$A := p \mid A \to A \mid \bot \mid \Box A \mid [j]A,$$

where $p \in V$ and $j \in Ag$. Formulas of the type $\Box A$ and $[j]A$ are informally read as 'A is (historically) necessary' and 'the agent j sees to it that A', respectively.

If A is a stit formula, then we denote by $|A|$ the set of propositional variables occurring in A and we denote by $Ag(A)$ the set of cstit modalities occurring in A.

Formulas of $\mathcal{L}^{Ag}_V$ are interpreted via the standard semantics of BT+AC structures as defined in, e.g., [2]. More precisely, an (Ag, V)-*stit model* is a structure of the form $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, such that:

- $Tree$ is a non-empty set. Elements of $Tree$ are called *moments*.
- $\leq$ is a partial order on $Tree$ for which a temporal interpretation is assumed.
- $Hist(Tree, \leq)$ is the set of maximal chains in $Tree$ w.r.t. $\leq$. Since $Hist(Tree, \leq)$ is completely determined by $Tree$ and $\leq$, it is not included into the structure of a model as a separate component. Elements of

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$Hist(Tree, \leq)$ are called *histories*. The set of histories containing a given moment m will be denoted $H^{\mathfrak{S}}_m$. The following set

$$MH(Tree, \leq) = \{(m, h) \mid m \in Tree,\ h \in H^{\mathfrak{S}}_m\},$$

called the set of *moment-history pairs*, will be used to evaluate formulas in $\mathcal{L}^{Ag}_V$.

- $Choice$ is a function mapping $Tree \times Ag$ into $2^{2^{Hist(Tree,\leq)}}$ in such a way that for any given $j \in Ag$ and $m \in Tree$ we have as $Choice(m, j)$ (to be denoted as $Choice^m_j$ below) a partition of $H^{\mathfrak{S}}_m$. For a given $h \in H^{\mathfrak{S}}_m$ we will denote by $Choice^m_j(h)$ the element of the partition $Choice^m_j$ (otherwise called a *choice cell*) containing h.
- V is an evaluation function, mapping the set V into $2^{MH(Tree,\leq)}$

In what follows, for a given (Ag, V)-stit model $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, we will sometimes use $Hist(\mathfrak{S})$ and $MH(\mathfrak{S})$ to denote $Hist(Tree, \leq)$ and $MH(Tree, \leq)$, respectively.

Additionally, every stit model $\mathfrak{S}$ is required to satisfy the following constraints:

(i) **Historical connection**:

$$(\forall m, m_1 \in Tree)(\exists m_2 \in Tree)(m_2 \leq m \,\&\, m_2 \leq m_1) \qquad \text{(HC)}$$

(ii) **No backward branching**:

$$(\forall m, m_1, m_2 \in Tree)((m_1 \leq m \,\&\, m_2 \leq m) \Rightarrow \\ \Rightarrow (m_1 \leq m_2 \vee m_2 \leq m_1)) \qquad \text{(NBB)}$$

(iii) **No choice between undivided histories**:

$$(\forall m \in Tree)(\forall h, h' \in H^{\mathfrak{S}}_m)((\exists m' > m)(h,h' \in H^{\mathfrak{S}}_{m'}) \Rightarrow \\ \Rightarrow Choice^m_j(h) = Choice^m_j(h')) \qquad \text{(NCUH)}$$

for every $j \in Ag$.

(iv) **Independence of agents**:

$$(\forall f : Ag \to 2^{H^{\mathfrak{S}}_m})((\forall j \in Ag)(f(j) \in Choice^m_j) \Rightarrow \bigcap_{j \in Ag} f(j) \neq \emptyset) \qquad \text{(IA)}$$

for every $m \in Tree$.

The inductive definition of the satisfaction relation for the members of $\mathcal{L}^{Ag}_V$ is then as follows:

$$\mathfrak{S}, m, h \models p \Leftrightarrow (m, h) \in V(p);$$
$$\mathfrak{S}, m, h \models [j]A \Leftrightarrow (\forall h' \in Choice^m_j(h))(\mathfrak{S}, m, h' \models A);$$
$$\mathfrak{S}, m, h \models \Box A \Leftrightarrow (\forall h' \in H^{\mathfrak{S}}_m)(\mathfrak{S}, m, h' \models A),$$

with the usual clauses for the Boolean connectives. The notions of satisfaction and validity are also defined in a standard way.

Under this semantics, the logic allows for a strongly complete Hilbert-style axiomatic system (see, e.g., [1] and also [2, Ch. 17] for an axiomatization of a somewhat more expressive logic). For a stit formula A, we will write $\vdash A$ to mean that A is provable in this axiomatic system. However, the question of interpolation and related properties for this logic remains (to the best of our knowledge) unexplored.

Since we are considering, in fact, an infinite family of languages which differ from one another in their sets of allowed modalities, it is natural to allow the number of agents in Ag as a parameter in our definitions of interpolation-related properties for stit logic. In this way, we first define the property of restricted interpolation for the stit logic of n agents:

Definition 0.1 For a positive integer n, stit logic has the *Restricted n-Craig Interpolation Property* (abbreviated by $(RCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}_V^{\{1,\dots,n\}}$, whenever $\vdash A \to B$ and $Ag(A) \cap Ag(B) = \emptyset$, then there exists a $C \in \mathcal{L}_{|A|\cap|B|}^{Ag(A)\cup Ag(B)}$ such that both $\vdash A \to C$ and $\vdash C \to B$.

Our main result then says that:

Theorem 0.2 *For every positive integer n, stit logic has $(RCIP)_n$ iff $n \leq 3$.*

The positive part is proved by using the standard technique based on constructing a canonical model serving as a counterexample for a given valid implication $\vdash A \to B$ on the assumption that no interpolant for it exists. However, ensuring that such a model satisfies (IA) turns out to be a non-trivial task. The fact that it can be carried out for $n \leq 3$ is due to the circumstance that the triple is the biggest group that cannot be divided into two parts without one of those parts being either empty or a singleton.

The negative part of the result is obtained by showing that the valid implication $\vdash \Diamond([j_1]p \wedge [j_2](p \to q)) \to \neg\Diamond([j_3]r \wedge [j_4](r \to \neg q))$, for pairwise different $j_1, j_2, j_3, j_4 \in Ag$, does not have an interpolant required by Definition 0.1. This is achieved by constructing a pair of bisimilar (in an appropriate sense) models for the community of four agents, one of which verifies the antecedent of the above implication, whereas the other one falsifies the consequent. These models exploit the sets of vertices of a four-dimensional cube.

Restricted interpolation is a weakening of Craig Interpolation property in that it only requires the existence of an interpolant when the sets of cstit modalities of the premise and the conclusion are disjoint. The respective definition of Craig Interpolation Property for stit logic of n agents can be given as follows:

Definition 0.3 Stit logic has the *n-Craig Interpolation Property* (abbreviated by $(CIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}_V^{\{1,\dots,n\}}$, whenever $\vdash A \to B$, then there exists a $C \in \mathcal{L}_{|A|\cap|B|}^{Ag(A)\cup Ag(B)}$ such that both $\vdash A \to C$ and $\vdash C \to B$.

An obvious corollary of our main result says that its negative part also holds for the more classical, unrestricted version of interpolation:

Corollary 0.4 *For all $n > 3$, stit logic does not have $(CIP)_n$.*

In modal logic, it is also common to consider a strengthening of Craig Interpolation Property which requires the interpolant to only contain modalities occurring both in the premise and in the conclusion. In the case of stit logic, it is natural to require this only for cstit modalities and not for the historical necessity. In this way, we obtain the following definitions for unrestricted and restricted versions of the interpolation property:

Definition 0.5 Stit logic has the *Strong Restricted n-Craig Interpolation Property* (abbreviated by $(SRCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}_V^{\{1,\dots,n\}}$, whenever $\vdash A \to B$ and $Ag(A) \cap Ag(B) = \emptyset$, then there exists a $C \in \mathcal{L}_{|A| \cap |B|}^{\emptyset}$ such that both $\vdash A \to C$ and $\vdash C \to B$.

Definition 0.6 Stit logic has the *Strong n-Craig Interpolation Property* (abbreviated by $(SCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}_V^{\{1,\dots,n\}}$, there exists a $C \in \mathcal{L}_{|A| \cap |B|}^{Ag(A) \cap Ag(B)}$ such that both $\vdash A \to C$ and $\vdash C \to B$.

This strengthening of interpolation notion leads to the following extension of the scope of interpolation failure as compared to Theorem 0.2:

Theorem 0.7 *For every positive integer $n > 1$, stit logic fails both $(SRCIP)_n$ and $(SCIP)_n$.*

The latter theorem is proved in the same way as the negative part of Theorem 0.2, however, the valid implication admitting of no interpolant is in this case much simpler and looks as follows: $\vdash \Diamond[j_1]p \to \neg\Diamond[j_2]\neg p$, where, again, it is assumed that j_1 and j_2 are different.

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Frame Definability and Extensions of First-Order Modal Logic

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Abstract

We will study frame definability for first-order modal logic and its extensions, specially for first-order hybrid logics, and give some versions of the Goldblatt-Thomason theorem for these logics.

Keywords: Frame definability, First-order modal logic, Firs-order hybrid logic, Goldblatt-Thomason theorem.

1 Introduction

Correspondence theory is one of the interesting issues in model theory of modal logics that always has been considered. In 1974, Goldblatt and Thomason gave one of the most important results in this area which is a characterization of elementary classes of frames definable by a set of propositional modal formulas. They proved that an elementary class of Kripke frames is definable by a set of propositional modal formulas if and only if it is closed under bounded morphic images, generated subframes and disjoint unions and reflects ultrafilter extensions [1].

This issue is widely studied for various extensions of modal language, for example, [4] for modal language with global modality and [3] for hybrid language.

In [8] we studied frame definability for first-order modal logic (FML) and gave a version of the Goldblatt-Thomason theorem for this logic. The advantage of this result, compared to the original Goldblatt-Thomason theorem, is

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that it does not need the condition of ultrafilter reflection and uses only closure under bounded morphic images, generated subframes and disjoint unions.

One question that can be raised here is finding the appropriate versions of the Goldblatt-Thomason theorem for extensions of first-order modal language. Here, we investigate this problem for FML with global modality and also for first-order hybrid languages. We show that, similar to modal logic most of first-oder case of these languages define a wider range of classes of frames compare to their propositional case. Although first-order hybrid language with satisfaction and binder operators does not define any larger class of Kripke frames than its propositional case.

2 Preliminaries

Throughout this paper, for simplicity, we consider first-order language without function symbols. So, assume that τ is a language containing countably many predicate and constant symbols. First-order modal formulas are defined by the following grammar:

$$FML := P(t_1, \dots, t_n) \mid t_1 = t_2 \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\exists x\phi) \mid (\Diamond\phi),$$

where P is an n-ary predicate and $t_1, \dots, t_n$ are terms (i.e. are constant or variable). There are many extensions of modal logic. Here is a list of those extensions we will study in this paper. Let Nom and $Svar$ be countable sets of nominals and state variables, respectively.

$$\begin{aligned} FML(E) &:= FML \mid E\phi, \\ FHL(E) &:= FML \mid i \mid E\phi, \\ FHL(@) &:= FML \mid i \mid @_i\phi, \\ FHL(@,\downarrow) &:= FHL(@) \mid s \mid @_s\phi \mid \ \downarrow s\phi, \end{aligned}$$

where $i \in Nom$ and $s \in Svar$. E is an existential modality and @ and $\downarrow$ respectively denote the satisfaction and binder operators in hybrid logic. (By PML we mean a propositional modal logic, similarly for PML(E), ..., PHL($@,\downarrow$).)

A *constant domain model* $\mathfrak{M}$ based on a Kripke frame $\mathfrak{F} = (W, R)$ is a tuple $\mathfrak{M} = (\mathfrak{F}, D, I)$, where $D \neq \emptyset$ is a domain and I is an interpretation function. It means that for each $w \in W$ and each n-ary predicate symbol P we have $I(w, P) \subseteq D^n$ and also $I(w, c) = I(w', c) \in D$, for each $w, w' \in W$ and each constant c.

A Kripke model $\mathfrak{M}$ is a model for hybrid logic if it has an interpretation for nominals, i.e. $I(i)$ is a unique element of W for each $i \in Nom$.

The satisfaction relation, $\mathfrak{M}, w \models_\sigma \phi$, is defined by the standard rules (see [2] for more details).

We say that a formula $\phi(x_1, \dots, x_n)$ is valid at a world w of the frame $\mathfrak{F}$ if for every constant domain model $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M}, w \models \forall x_1 \dots \forall x_n \phi(x_1, \dots, x_n)$. We also say $\phi(x_1, \dots, x_n)$ is valid on the frame $\mathfrak{F}$ whenever it is valid in every world $w \in W$.

Let L be any of the above logical language. We say that a class of Kripke frames K is *L-definable* if there exists a set of L-sentences Λ such that for any frame $\mathfrak{F}$, we have $\mathfrak{F} \in K$ if and only if all sentences of Λ are valid on $\mathfrak{F}$.

It is easy to see that if a class of Kripke frames K is PML-definable then it is FML-definable, similarly for other above logical languages. But the converse of this fact is not true in general. In the following example we show that there is a class of frames which is FML-definable but is not PML-definable.

Example 2.1 Consider the class of frames in which every world has a reflexive accessible world (i.e. $\forall x \exists y(Rxy \wedge Ryy)$). Although this class is not definable by any set of propositional modal formulas, because it does not reflect ultrafilter extensions, it is definable by the formula $\Diamond \forall x(\Box P(x) \to P(x))$.

3 Main Results

The following theorem is a version of the Goldblatt-Thomason theorem in FML.

Theorem 3.1 *[8] Let K be an elementary class of frames. Then K is FML-definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions.*

Now we investigate the Goldblatt-Thomason theorem for extensions of FML.

In [7] van Benthem showed that an elementary class of Kripke frames is definable in propositional modal logic with global modality if and only if it is closed under bounded morphic images and reflects ultrafilter extensions. By using the same method from theorem 3.1 we can prove that:

Proposition 3.2 *An elementary class K of Kripke frames is definable by a set of FML(E)-sentences if and only if it is closed under bounded morphic images.*

In hybrid logics $PHL(E)$ and $PHL(@)$, instead of ultrafilter extensions the notion of ultrafilter morphic images plays the essential rule. Ten Cate in [3] studied the frame definability for hybrid logics. He showed that an elementary class of frames is definable by a set of PHL(@)-formulas if and only if it is closed under ultrafilter morphic images and generated subframes. Also, he proved that an elementary class K is PHL(E)-definable if and only if it is closed under ultrafilter morphic images. Now we can adjust the proof of the Goldblatt-Thomason for FML to prove the following results which show that the closure of ultrafilter morphic images is no longer valid for FHL(@) and FHL(E) sentences.

Theorem 3.3 *Let K be an elementary class of frames. Then K is definable by a set of FHL(@)-sentences if and only if K is closed under generated subframes.*

Proof. (Sketch) We want to show that the set Λ_K of all valid sentences on all frames of K defines K. The non-trivial direction is to show that if $\mathfrak{F} \models \Lambda_K$, then $\mathfrak{F} \in K$.

Let τ' be a language containing a one-ary predicate P, a two-ary predicate R, a constant c_w and nominal i_w for each world $w \in W$ and also a k-ary predicate $R_\theta(x_1, \ldots, x_k)$ for each $\mathcal{L}_R = \{R, =\}$-formula $\theta(x_1, \ldots, x_k)$.

Now construct a τ'-model $\mathfrak{M}$ based on $\mathfrak{F}$ by setting $D_{\mathfrak{M}} = W_{\mathfrak{F}}$ and naturally interpreting τ' symbols in a way that the theory of $(\mathfrak{M}, w)$ describes the accessibility relation of $\mathfrak{F}$ uniformly in all worlds. In other words for each $w \in W_{\mathfrak{F}}$ and each predicate R_θ we have

$$I_{\mathfrak{M}}(w, R_\theta) = \{(w_1, \ldots, w_k) \in D^k \mid \mathfrak{M} \models \theta(w_1, \ldots, w_k)\}.$$

Let $\Delta = \{@_{i_w}\psi \mid \psi \text{ is a } \tau' - \text{sentence and } \mathfrak{M} \models @_{i_w}\psi\}$.

Δ is finitely satisfiable in K, so there is a model $(\mathfrak{N}, v)$ based on some frame $\mathfrak{G} \in K$ such that $\mathfrak{N}, v \models \Delta$.

Since K is closed under generated subframes we can assume that $\mathfrak{G}$ is generated by $\{I_{\mathfrak{N}}(i_w) \mid w \in W_{\mathfrak{F}}\}$.

Contrary to the proof of the Goldblatt-Thomason theorem for FML, we do not need to extract a frame from the domain of $\mathfrak{N}$ which is a bounded morphic image of $\mathfrak{G}$ and an $\{R, =\}$-elementary extension of $\mathfrak{F}$. In here we can prove that the function $f : W_{\mathfrak{F}} \to W_{\mathfrak{G}}$ defined as $f(w) = I_{\mathfrak{N}}(i_w)$ is an elementary embedding. □

Corollary 3.4 *If K is an elementary class of frames, then K is FHL(E)-definable.*

Clearly the converse of the above theorem does not hold, it suffices to consider some properties which are definable in PML but are not elementary. So, Corollary 3.4 shows that FHL(E) has a more expressive power than first-order language in describing relational properties.

But what about FHL$(@, \downarrow)$? As we see in Example 2.1, the class of frames in which every world has a reflexive successor is FML-definable though is not PML-definable. However, this class is defined in PHL$(@, \downarrow)$ by the sentence $\Diamond \downarrow s.\Diamond s$.

In [3], it is shown that an elementary class of frames (1) is definable in PHL$(@, \downarrow)$ iff (2) it is definable by a single pure [3] PHL$(@, \downarrow)$-sentence iff (3) it is closed under generated subframes and reflects finitely generated subframes.

Since definability by pure sentences is not related to be in PHL$(@, \downarrow)$ or FHL$(@, \downarrow)$ and the clause (1) implies (3) in first-order case too, we can conclude that:

Theorem 3.5 *An elementary class of frames is PHL$(@, \downarrow)$-definable if and only of it is FHL$(@, \downarrow)$-definable.*

So, FHL$(@, \downarrow)$ does not have more expressive power to define frame properties than its propositional case.

4 Future Research

There are many questions raised from these partial results. For example:

Find an alternative proof for the Goldblatt-Thomason theorem for FML and its extensions for varying domain models.

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[3] Pure formula is a formula without any propositional variable.

Provide a precise categorization of these logical language. For instance, give some conditions under which a FHL($@, \downarrow$)-definable class of Kripke frames (not necessarily elementary) is FML-definable.

Similar to propositional modal logic, the Goldblatt-Thomason theorem for FML is given for elementary classes of frames and the question of finding an alternative result without this restriction remains open. However, in [4] van Benthem gave versions of the Goldblatt-Thomason theorem for (not necessarily elementary) classes of finite transitive frames and also for finite frames. Both of these theorems hold for FML too.

A (constant or varying) domain *skeleton* is a triple (W, R, D) where (W, R) is a Kripke frame and D is a domain. The Goldblatt-Thomason theorem also could be studied with respect to class of skeletons. Each skeleton could be considered as a first-order structure in the language $\{R, E, =\}$ where $\{R, =\}$ is a frame language and E is a predicate indicates which objects belong to which worlds. In [6] (and later in [5]) van Benthem showed that any first-order formula α without object variable in the language $\{R, =\}$ has a corresponding formula in FML whenever it has a corresponding in PML. So, for any class of skeletons K definable by a set of first-order $\{R, =\}$-sentences, K is FML-definable if and only if it is PML-definable. He also gave an example which is FML-definable but is not first-order definable.

But one can ask this question for a class of skeletons defined by a fist-order formulas containing the predicate E. For example the class of increasing or decreasing domain skeletons which are FML-definable by the following sentences:

$$\text{(Barcan formula.)} \quad \forall x \Box \phi(x) \to \Box \forall x \phi(x)$$

$$\text{(Converse of Barcan formula.)} \quad \Box \forall x \phi(x) \to \forall x \Box \phi(x)$$

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Compactness for Modal Probability Logic

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Abstract

In this talk we study the compactness property for probability logic and present some results about some compact fragments of this logic with respect to σ-additive and finitely additive type spaces.

Keywords: Modal probability logic, Type spaces, Compactness theorem, Ultraproduct.

1 Introduction and Preliminaries

Probabilistic systems are widely used in theoretical computer science and game theory. In 1967, Harsanyi introduced the notion of type spaces, which provides an implicit description of beliefs in games with incomplete information played by Bayesian players. A type space is a measurable space equipped with a type function, a function that for each agent assigns to each world a probability measure which express the degree of beliefs of agents. In type spaces the type function is a σ-additive measure but in some applications such as in decision theory it is useful to consider finitely additive type functions.

The syntax used in this paper is based on the one introduced by Aumann in [1]. It is obtained by adding countably many belief operators L_r, for every rational $r \in [0, 1]$, to propositional logic. The operator $L_r\phi$ means that "the agent assigns probability at least r to the event ϕ" [3]. So, probability formulas are defined by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid L_r\phi, \quad (r \in \mathbb{Q} \cap [0, 1])$$

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[3] For simplicity throughout this paper we just consider one agent probability logic.

Also one can derive the operator M_r means that "the agent assigns probability at most r" as follows:

$$M_r\phi = L_{1-r}\neg\phi.$$

A probability model (or a type space) is a tuple $\mathfrak{M} = (\Omega, \mathcal{A}, T, V)$ where

- $(\Omega, \mathcal{A})$ is a measurable space,
- T is a measurable function from Ω to the measurable space $\Delta(\Omega, \mathcal{A})$ of probability measures on Ω with the σ-algebra generated by the sets $\{\mu \in \Delta(\Omega, \mathcal{A}) \mid \mu(E) \geq r\}$ for all $E \in \mathcal{A}$ and $r \in \mathbb{Q} \cap [0, 1]$.
- v is a valuation function such that $v(p) \in \mathcal{A}$ for each $p \in \mathcal{P}$.

The satisfaction relation is defined inductively in the usual way for propositional variables and boolean connectives and

$$\mathfrak{M}, w \models L_r\phi \quad \text{if and only if} \quad T(w)(\llbracket\phi\rrbracket_{\mathfrak{M}}) \geq r$$

where $\llbracket\phi\rrbracket_{\mathfrak{M}} = \{w' \in \Omega \mid \mathfrak{M}, w' \models \phi\}$. Based on the definition of probability model, it is easy to see that $\llbracket\phi\rrbracket_{\mathfrak{M}} \in \mathcal{A}$ for every formula ϕ. Also, by definition of M_r we have:

$$\mathfrak{M}, w \models M_r\phi \quad \text{if and only if} \quad T(w)(\llbracket\phi\rrbracket_{\mathfrak{M}}) \leq r.$$

We often omit the subscript $\mathfrak{M}$ and write $\llbracket\phi\rrbracket$ when no confusion can arise.

The axiom systems and the soundness and completeness theorems were studied for probability logic (with respect to both σ-additive and finitely additive type spaces) in [2,4,3,5]. Also it is known that probability logic is not compact with respect to σ-additive or finitely additive type spaces. To see this consider the set

$$\{L_{\frac{1}{2}-\frac{1}{2^{n+1}}}p \mid n \in \mathbb{N}\} \cup \{\neg L_{\frac{1}{2}}p\}$$

which is finitely satisfiable and is not satisfiable. The non-compactness of this kind of examples is due to the existence of formulas in the form $\neg L_r$ (or $\neg M_r$) expressing the strict inequality. So, one may wonder whether the compactness theorem holds for those fragments of probability logic can not describe strict inequalities $<$ and $>$.

In this talk we introduced two fragments of probability logic which can express just the inequality $\geq$ or both of $\leq, \geq$ and investigate whether the compactness property holds for this fragments with respect to class of σ-additive or finitely-additive type spaces. So, let

$$\mathcal{L}^+ := p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid L_r\phi,$$
$$\mathcal{L}^+_{\leq} := \mathcal{L}^+ \mid M_r\phi,$$

where p is a proposition and $r \in \mathbb{Q} \cap [0, 1]$. Note that neither of the strict inequalities $<, >$ are expressible in both of these fragments.

2 The Compactness Theorem

As mentioned in Section 1, probability logic is not compact. Now consider the following example:

Example 2.1 Let

$$\Sigma = \{\ M_0(M_0 p \vee L_1 p)\ \} \cup \{\ M_{\frac{1}{2}}(L_{\frac{1}{2^i}} p \wedge M_{1-\frac{1}{2^i}} p) \mid i \in \mathbb{N}\}.$$

Σ is finitely satisfiable but it is not satisfiable in any probability model. However, it has a finitely additive model $\mathfrak{M} = (\mathbb{N}, \mathcal{P}(\mathbb{N}), T, v)$ where $v(p) = \{0\}$ and the function T is defined as follows:

- For each $n \neq 0$,

$$T(n)(\{x\}) = \begin{cases} \frac{1}{2^n} & \text{if } 0 \leq x \leq 2^n - 1 \\ 0 & \text{if } x > 2^n - 1 \end{cases}$$

- For $n = 0$, let U be a non-principal ultrafilter over $\mathbb{N}$ and define

$$T(0)(X) = \begin{cases} 1 & \text{if } X \in U \\ 0 & \text{if } X \notin U \end{cases}$$

Then, $\mathfrak{M}, 0 \models \Sigma$.

Due to the referee's comment, the above example also holds for the simpler set

$$\{M_0 M_0 p\} \cup \{L_1 M_{\frac{1}{2^n}} p \mid n \in \mathbb{N}\}.$$

The above example shows that even with restricting to $\mathcal{L}^+_{\leq}$ the compactness property does not hold with respect to σ-additive type spaces. But, since this example has a finitely additive model, one may wonder whether the compactness property holds for wider range of formulas with respect to finitely additive models.

One of the techniques of proving the compactness property in model theory is based on using the ultraproducts. Here we introduce the notion of ultraproducts of type spaces and show that $\mathcal{L}^+$ is a compact fragment of probability logic.

Definition 2.2 Let $\langle \mathfrak{M}_i = (\Omega_i, \mathcal{A}_i, T_i, v_i) : i \in I \rangle$ be a family of probability models and U be a non-principal ultrafilter over I. The ultraproduct of $\mathfrak{M}_i$s over U is a model $\mathfrak{M} = \prod_U \mathfrak{M}_i = (\Omega_U, \mathcal{A}_U, T_U, v_U)$ where

- $\Omega_U = \prod_U \Omega_i$,
- $\mathcal{A}_U$ is a σ-algebra generated by a boolean algebra $\mathcal{A}$, the set of all $[(A_i)]$s where $A_i \in \mathcal{A}_i$ and $[(A_i)] = \{[(a_i)] \in \Omega_U \mid \{i \in I \mid a_i \in A_i\} \in U\}$,
- T_U is a measurable function induced by $T' : \prod_U \Omega_i \times \mathcal{A} \to [0, 1]$ defined as

$$T'([(w_i)])([(A_i)]) = \lim_U T_i(w_i)(A_i).$$

- $[(w_i)] \in v_U(p)$ if and only if $\{i \in I \mid w_i \in v_i(p)\} \in U$.

The Łoś theorem does not hold in general for any probability formula ϕ. But we have

Theorem 2.3 *For a family of probability models $\langle \mathfrak{M}_i : i \in I \rangle$ and a non-principal ultrafilter U over I, we have*

$$\text{if } \{i \in I \mid \mathfrak{M}_i, w_i \models \phi\} \in U, \text{ then } \prod_U \mathfrak{M}_i, [(w_i)] \models \phi,$$

for every formula $\phi \in \mathcal{L}^+$.

So we can conclude that:

Theorem 2.4 ($\mathcal{L}^+$-Compactness) *Suppose that Γ is a set of $\mathcal{L}^+$-formulas. Γ has a σ-additive model provided that it is finitely satisfiable.*

One of the other methods of proving the compactness theorem in model theory is a Henkin construction. By this methods, which is also used in [5] to prove the strong completeness of probability logic with respect to finitely additive type spaces, we show that $\mathcal{L}^+_{\leq}$ is a compact fragment of probability logic with finitely additive models.

Theorem 2.5 ($\mathcal{L}^+_{\leq}$-Compactness) *Let Γ be a set of $\mathcal{L}^+_{\leq}$-formulas which is finitely satisfiable. Then Γ has a finitely additive model.*

Proof. (Sketch) Suppose that $\psi_1, \psi_2, \dots$ is an enumeration of $\mathcal{L}^+_{\leq}$-formulas. Put $\Sigma_0 = \Gamma$. For each $n \in \mathbb{N}$ define

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\psi_{n+1}\} & \text{if it is finitely satisfiable} \\ \Sigma_n & \text{otherwise.} \end{cases}$$

Let $\Sigma = \bigcup \Sigma_n$, it is easy to see that Σ is maximal finitely satisfiable. Also for each formula $\phi \in \mathcal{L}^+_{\leq}$ and each $r \in \mathbb{Q} \cap [0,1]$ the set Σ contains at least one of the formulas $L_r\phi$ and $M_r\phi$.

Then we construct a model $(\mathfrak{M}, w_0)$ in a way that $\mathfrak{M}, w_0 \models \Sigma$.

Let Ω be the set of all maximal finitely satisfiable sets of $\mathcal{L}^+_{\leq}$-formulas. Put Θ be the set of all $[\phi] = \{w \in \Omega \mid \phi \in w\}$, for each formula $\phi \in \mathcal{L}^+_{\leq}$. Θ is a lattice.

Now define the function $T' : \Omega \times \Theta \to [0,1]$ as follows:

$$T'(w)([\phi]) = \sup\{r \mid L_r\phi \in w\}.$$

For each $w \in \Omega$ the function $T'(w)()$ is a valuation on the lattice Θ. So we can extend it to a finitely additive measure $T(w)()$ on $\mathcal{P}(\Omega)$.

Finally for each proposition p, put $v(p) = \{w \in \Omega \mid p \in w\}$.

Now by induction on the complexity of formulas we can show that for every formula $\phi \in \mathcal{L}^+_{\leq}$ and every $w \in \Omega$,

$$\text{if } \phi \in w, \text{ then } \mathfrak{M}, w \models \phi.$$

$\square$

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Informational semantics for superintuitionistic modal logics

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Abstract

In this paper we present a non-standard semantic framework that generalizes Kripke semantics in two directions: (i) it allows for weakening of some laws for "normal" modalities and (ii) it allows to weaken the background non-modal propositional logic to intuitionistic logic. In other words, the framework can be seen as a novel semantics for superintuitionistic non-normal modal logics. A peculiar feature of the framework is that not only diamond but also box is captured as an existential modality. We will show how this framework incorporates standard neighborhood semantics and indicate that it incorporates in a similar sense Božić and Došen's semantics for normal intuitionistic modal logics.

Keywords: Intuitionistic modal logic, neighborhood semantics, information models, information states.

1 Introduction

Kripke semantics, as presented for example in [2], is the most well-known and paradigmatic semantic framework for modal logic. Several generalizations of this framework have been proposed in the literature. For example, neighborhood semantics [7] allows for weakening of some laws for "normal" modalities that are inevitable in standard Kripke semantics but the background non-modal propositional logic is preserved. The framework from [3] illustrates another type of generalization of Kripke semantics in which the "normal" rules for modalities are preserved but the background non-modal propositional logic is weakened.

In this paper we intend to present a non-standard semantic framework for superintuitionistic modal logics that we call *informational semantics*. Our aim is to argue that informational semantics provides not only a novel perspective on modal logic (which is interesting on its own) but it also generalizes Kripke semantics in both mentioned directions and is flexible enough to incorporate neighborhood semantics as well as the framework from [3].

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A peculiar feature of the framework is that not only diamond but also box is captured as an existential modality. This is inspired by [1,4], where reasons for an existential interpretation of box-like modalities are elaborated. Our framework is related to (even though not identiacal with) several frameworks that have already appeared in the literature on modal logic. In particular, there are similarities with the frameworks presented in [6,11,5,10,12]. [3]

2 Informational semantics for intuitionistic logic

The non-modal part of the semantics was already introduced and explored in [8,9]. The primitive entities of the framework are called *information states.* We will assume that for any two states a and b there is a state $a \circ b$ that can be called the *common content* of a and b. Moreover, we will assume that there is also an *inconsistent state* i.

Definition 2.1 An *information frame* (IF) is a distributive semilattice with a neutral element, i.e. an algebraic structure $\langle S, \circ, i\rangle$, where S is a non-empty set, $\circ$ is a commutative, associative and idempotent binary relation with respect to which i is a neutral element (i.e. $a \circ i = a$ for any $a \in S$), and satifying the following distributivity condition: $c \sqsubseteq a + b$ implies $\exists d, e : d \sqsubseteq a,\ e \sqsubseteq b$ and $d \circ e = c$, where $a \sqsubseteq b$ is defined as $a \circ b = b$. [4]

An essential feature of information states is that they support pieces of information. So, the next step is to define relation of support ($\Vdash$) between information states and formulas. As is usual in various relational semantics, this relation is stipulated for atomic formulas (via valuation) and consequently recursively defined for complex formulas. The informal interpretation of i and $\circ$ leads to the following constraints: (a) the inconsistent state i supports p; (b) the common content of a and b (i.e. $a \circ b$) supports p iff both states a and b support p. In algebraic terms we will assume that the set of states that support an atomic formula forms an ideal in the join-semilattice.

Definition 2.2 An *information model* (IM) is a pair $\langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is an IF and V is a valuation in $\mathcal{F}$, i.e. a function that assigns to every atomic formula an ideal in $\mathcal{F}$.

Given an IM, the support relation between formulas of a propositional non-modal language and information states are defined in the following way: $a \Vdash p$ iff $a \in V(p)$; $a \Vdash \bot$ iff $a = i$; $a \Vdash \varphi \to \psi$ iff $\forall b \sqsubseteq a$, if $b \Vdash \varphi$, then $b \Vdash \psi$; $a \Vdash \varphi \wedge \psi$ iff $a \Vdash \varphi$ and $a \Vdash \psi$; $a \Vdash \varphi \vee \psi$ iff $\exists b, c$: $b \Vdash \varphi$, $c \Vdash \psi$ and $a = b \circ c$.

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[3] Due to the lack of space, we cannot provide a more careful comparison here.

[4] Information states are primitive entities of our framework and they are not required to have any internal structure but, informally speaking, they can be intuitively viewed in two dual ways: (i) as sets of sentences or (ii) as sets of possible worlds. Regarded from the former point of view, it is natural to interpret $\circ$ as meet (corresponding to intersection) and i as the top element. Dually, viewed from the latter perspective, $\circ$ can be naturally regarded as join (corresponding to union) and i as the bottom element. Officially, we take the second perspective and regard the structures as distributive join-semilattices, where i is the least element.

It turns out that complex formulas, like atomic formulas, express ideals in the IM (distributivity of the semilattice is needed for this result). A formula is said to be valid in an information model iff it is supported by every state of that model. A formula is valid in a class of information models iff it is valid in every model of that class. A formula is valid in a class of information frames iff it is valid in any model on any frame from that class.

Proposition 2.3 *The set of formulas valid in all IFs is exactly the set of intuitionistically valid formulas.*

The main aim of our paper is to extend this semantics with modalities.

3 Modalities

As already mentioned a peculiar feature of our framework is that box and diamond are both existential modalities. So they are not distinguished by different semantic clauses but, as we will see, the respective clauses are relative to two different accessibility relations.

Definition 3.1 A *modal information frame* (MIF) is a structure $\langle S, \circ, R_\Diamond, R_\Box, i\rangle$ such that $\langle S, \circ, i\rangle$ is an IF and $R_\Diamond, R_\Box$ are binary relations on S both satisfying the following conditions: (a) iRi; (b) if aRb and $c \sqsubseteq a$, then cRb; (c) if aRb and cRd, then $(a \circ c)R(b \circ d)$. A MIF equipped with a valuation is called a *modal information model* (MIM).

The support conditions for $\Diamond$ and $\Box$ are:

- $a \Vdash \Diamond\varphi$ iff there is b such that $aR_\Diamond b$ and $b \Vdash \varphi$.
- $a \Vdash \Box\varphi$ iff there is b such that $aR_\Box b$ and $b \Vdash \varphi$.

It holds for every formula of the modal language and any MIM that the set of its states supporting the formula forms an ideal in the MIM.

Proposition 3.2 *The set of formulas valid in all MIFs can be axiomatized by an axiomatization of intuitionistic logic enriched with two rules for modalities:*

$$M_\Diamond \ \varphi \to \psi / \Diamond\varphi \to \Diamond\psi \qquad\qquad M_\Box \ \varphi \to \psi / \Box\varphi \to \Box\psi$$

This logic will be called *the basic modal intuitionistic logic* (B-MIL). In this basic logic, the two modalities behave in the same way. That is, box does not have any specific features of the usual box-like modalities and the same holds for diamond. For this reason we will be interested in the extensions of B-MIL and their semantic representation within this framework.

Definition 3.3 A set of formulas is called a *logic* if it contains all axioms of intuitionistic logic and is closed under uniform substitution, modus ponens, and the rules $M_\Diamond$, $M_\Box$.

Let λ be a logic. One can construct a canonical model of λ as the MIM $\mathcal{M}^\lambda = \langle S, \circ, R_\Diamond, R_\Box, i, V\rangle$, where S is the set of all theories of λ; $\Gamma \circ \Delta$ is defined as $\Gamma \cap \Delta$; $\Gamma R_\Diamond \Delta$ iff for any $\varphi \in \Delta, \Diamond\varphi \in \Gamma$; $\Gamma R_\Box \Delta$ iff for any $\varphi \in \Delta, \Box\varphi \in \Gamma$; i is the set of all formulas; $\Delta \in V(p)$ iff $p \in \Delta$.

Proposition 3.4 *In any canonical model:* $\Delta \Vdash \varphi$ *iff* $\varphi \in \Delta$.

Definition 3.5 We say that a schema *strongly characterises* a class of MIFs $\mathcal{C}$ if $\mathcal{C}$ contains the canonical model of every logic that contains the schema, and for any MIF $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}$ iff all instances of the schema are valid in $\mathcal{F}$.

Strong characterization gives us completeness results in the following sense: Assume that we have an axiomatic system A that generates a logic corresponding to the set of formulas valid in a class of MIFs $\mathcal{C}$. Assume that S is a schema that strongly characterises a class of MIFs $\mathcal{D}$. Then the logic generated by the system A enriched with S generates a logic corresponding to the set of formulas valid in the class of MIFs $\mathcal{C} \cap \mathcal{D}$.

Proposition 3.6 *The schemata*

A_1 $\Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi)$.

A_2 $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$.

A_3 $\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Diamond\psi)$.

A_4 $(\Diamond\varphi \wedge \Box\psi) \rightarrow \Diamond(\varphi \wedge \psi)$.

A_5 $\varphi \rightarrow \Diamond\varphi$.

A_6 $\Box\varphi \rightarrow \varphi$.

A_7 $\Box\neg\varphi \rightarrow \neg\Diamond\varphi$ *(or equivalently* $\Diamond\neg\varphi \rightarrow \neg\Box\varphi$*).*

respectively strongly characterise the classes of frames defined by the following conditions:

C_1 $\forall a, b, c$: *if* $aR_\Diamond(b \circ c)$, *then* $\exists d, e : dR_\Diamond b,\ eR_\Diamond c,\ a = d \circ e$.

C_2 $\forall a, b, c$: *if* $aR_\Box b$ *and* $aR_\Box c$, *then* $\exists d$: $d \sqsubseteq b,\ d \sqsubseteq c,\ aR_\Box d$.

C_3 $\forall a, b, c$: *if* $aR_\Box(b \circ c)$, *then* $\exists d, e : dR_\Box b,\ eR_\Diamond c,\ a = d \circ e$.

C_4 $\forall a, b, c$: *if* $aR_\Diamond b$ *and* $aR_\Box c$, *then* $\exists d$: $d \sqsubseteq b,\ d \sqsubseteq c,\ aR_\Diamond d$.

C_5 $\forall a\ \exists b$: $aR_\Diamond b$ *and* $b \sqsubseteq a$.

C_6 $\forall a, b$: *if* $aR_\Box b$, *then* $a \sqsubseteq b$.

C_7 $\forall a, b, c$: *if* $a \neq i$, $aR_\Box b$ *and* $aR_\Diamond c$, *then* $\exists d : d \neq i,\ d \sqsubseteq b$ *and* $d \sqsubseteq c$.

Modal information models generalize monotone neighborhood models in the following sense. Let us recall that a neighborhood model is a triple $\langle W, N, V\rangle$, where $W \neq \emptyset$ is a set of possible worlds, $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ is a neighborhood function, and $V : At \rightarrow \mathcal{P}(W)$ is a valuation. The relation of truth ($\vDash$) between possible worlds and formulas is defined standardly for non-modal connectives and for modalities in the following way: $w \vDash \Box\varphi$ iff $|\varphi| \in N(w)$, and $w \vDash \Diamond\varphi$ iff $W - |\varphi| \notin N(w)$, where $|\varphi|$ is the set of worlds in which φ is true. A neighborhood model is monotone if $X \in N(w)$ and $X \subseteq Y$ implies $Y \in N(w)$. Given a monotone neighborhood model $\mathcal{M} = \langle W, N, V\rangle$ one can construct an information model $\mathcal{M}^* = \langle S, \circ, R_\Diamond, R_\Box, i, V^*\rangle$ as follows: $S = \mathcal{P}(W)$; $a \circ b = a \cup b$; $aR_\Diamond b$ iff $\forall w \in a : W - b \notin N(w)$; $aR_\Box b$ iff $\forall w \in a : b \in N(w)$; $i = \emptyset$; $a \in V^*(p)$ iff $a \subseteq V(p)$.

Proposition 3.7 *$\mathcal{M}$ and $\mathcal{M}^*$ validate the same formulas.*

The models of the framework for normal intuitionistic modal logics from [3] can be transformed into equivalent modal information models in a similar way.

4 Conclusion

In future research, we intend to develop a full-fledged semantic theory based on information models. For example, we will study such notions as bisimulation and model filtration in our framework. Moreover, we intend to apply this general framework to some special cases of superintuitionistic modal logics that turned out to be be especially complicated as, for example, modal Gödel-Dummett logic. We also want to exploit the fact that informational semantics can be extended in a straightforward way with inquisitive disjunction known form inquisitive logic [9], and the dependance atom from dependence logic [12]. Moreover, we intend to further generalize the framework to be applicable to a broad class of substructural modal logics.

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From the positive fragment of PDL to its non-classical extensions

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Abstract

We provide a complete binary implicational axiomatization of the positive fragment of Propositional Dynamic Logic, extending the work of Dunn [4] on positive modal logic. The intended application of this result are completeness proofs for non-classical extensions of positive PDL. Two examples will be outlined, namely, a paraconsistent extension with modal De Morgan negation and a substructural extension with the residual operators of the Non-associative Lambek calculus.

Keywords: Lambek calculus, dynamic logic, First Degree Entailment, paraconsistent logic, Propositional Dynamic Logic, substructural logics

1 Introduction

PDL [6,7] is a well known logic with applications in program verification, dynamic epistemic logic and deontic logic, for example. More generally, PDL can be seen as a logic of *structured actions* that modify various types of *objects*; be it programs modifying states of the computer, information state updates or actions of agents changing the world around them.

PDL is a normal modal logic and so it cannot represent some interesting types of objects and actions in a satisfactory way. Normal modal logics are closed under classical consequence; hence, the objects modified by the structured actions of PDL cannot represent non-trivially inconsistent information (because of the Explosion rule of classical logic, according to which every formula follows from an inconsistent set of premises) or states of agents with limited reasoning capabilities (agent have 'immediate' access to all consequences of their information), for example.

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This motivates the study of non-classical propositional dynamic logics (PDLs), an area of modal logic largely underdeveloped so far. [4]

We outline the first steps towards a general study of non-classical PDLs. We are ultimately interested in variants of PDL over non-classical extensions of the Distributive Lattice Logic DLL (defined below). Instead of studying such extensions individually, we focus on a PDL-style modal extension of DLL first (i.e. we extend the work of Dunn [4] on positive modal logic). Our completeness result for positive PDL can then be used as a building block in completeness proofs for various non-classical combinations of positive PDL with extensions of DLL (including paraconsistent logics, but also many distributive substructural logics). We discuss two examples in more detail, namely, a paraconsistent extension of positive PDL with modal De Morgan negation and a substructural extension with the residual operators of the Non-associative Lambek calculus.

2 Positive PDL

The *language* $\mathcal{L}$ contains two classes of expressions, namely, actions and formulas, defined by mutual induction:

$$\mathsf{Act} \quad A := a \mid A;A \mid A \cup A \mid A^* \mid X?$$

$$\mathsf{Form} \quad X := p \mid X \wedge X \mid X \vee X \mid [A]X \mid \langle A \rangle X$$

where $a \in \mathsf{AAct}$ (a countable set of 'atomic actions'), and $p \in \mathsf{Prop}$ (a countable set of propositional variables). For details on the informal interpretation of the language see [7, 164–167].

A *consequence $\mathcal{L}$-pair* is an ordered pair of formulas, written as $X \vdash Y$. A *dynamic model* is a couple $M = \langle W_M, [\![\ ]\!]_M \rangle$ where $W_M \neq \emptyset$ and

$[\![A]\!]_M$ is a binary relation on W_M and

$[\![X]\!]_M$ is a subset of W_M.

We assume the standard definitions of $[\![X]\!]_M$ and $[\![A]\!]_M$ for non-atomic formulas; see [7, 167–170]. $X \vdash Y$ is *valid in a model* M ($X \vdash_M Y$) iff $[\![X]\!]_M \subseteq [\![Y]\!]_M$; $PDL^+ = \{X \vdash Y \mid \forall M : X \vdash_M Y\}$. DLL comprises $X \vdash Y$ from PDL^+ that contain only $\wedge, \vee$. The binary implicational proof system (the terminology derives from [5]) $\mathcal{PDL}^+$ contains:

Axioms

$$X \vdash X \qquad X \wedge Y \vdash X \qquad X \wedge Y \vdash Y \qquad X \vdash X \vee Y \qquad Y \vdash X \vee Y$$

$$X \wedge (Y \vee Z) \vdash (X \wedge Y) \vee (X \wedge Z)$$

$$[A]X \wedge [A]Y \vdash [A](X \wedge Y) \qquad \langle A \rangle (X \vee Y) \vdash \langle A \rangle X \vee \langle A \rangle Y$$

$$[A](X \vee Y) \vdash [A]X \vee \langle A \rangle Y \qquad \langle A \rangle X \wedge [A]Y \vdash \langle A \rangle (X \wedge Y)$$

$$[A;B]X \dashv\vdash [A][B]X \qquad \langle A;B \rangle X \dashv\vdash \langle A \rangle \langle B \rangle X$$

$$[A \cup B]X \dashv\vdash [A]X \wedge [B]X \qquad \langle A \cup B \rangle X \dashv\vdash \langle A \rangle X \vee \langle B \rangle X$$

$$[A^*]X \dashv\vdash X \wedge [A][A^*]X \qquad X \vee \langle A \rangle \langle A^* \rangle X \dashv\vdash \langle A^* \rangle X$$

$$[Y?]X \wedge Y \vdash X \qquad Y \wedge X \vdash \langle Y? \rangle X \qquad [X?]X \vdash [A][X?]X \qquad Z \vdash X \vee [X?]Y$$

[4] For fuzzy PDLs, see [10,8,2,15]. A more general algebraic approach is [11]. Paraconsistent PDL has been explored in [13,14].

Rules

$$\frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z} \qquad \frac{X \vdash Y \quad X \vdash Z}{X \vdash Y \wedge Z} \qquad \frac{Y \vdash X \quad Z \vdash X}{Y \vee Z \vdash X}$$

$$\frac{X \vdash Y}{[A]X \vdash [A]Y} \qquad \frac{X \vdash Y}{\langle A\rangle X \vdash \langle A\rangle Y}$$

$$\frac{X \vdash [A]X}{X \vdash [A^*]X} \qquad \frac{\langle A\rangle X \vdash X}{\langle A^*\rangle X \vdash X} \qquad \frac{X \wedge Y \vdash Z}{X \vdash [Y?]Z}$$

The *canonical structure* is $C = \langle P, [\![\,]\!]_C\rangle$ where P is the set of all non-empty prime theories (sets of formulas closed under $\vdash_{\mathcal{PDL}^+}$ such that $X \vee Y$ is in the set only if X or Y is) and $[\![\,]\!]_C$ is a function such that

$[\![X]\!]_C = \{t\ ;\ X \in t\}$ and

$[\![A]\!]_C$ is a binary relation on P such that $t[\![A]\!]_C t'$ iff (i) for all X, $[A]X \in t$ only if $X \in t'$ and (ii) for all X, $X \in t'$ only if $\langle A\rangle X \in t$.

The canonical structure is *not* a dynamic model—although $[\![A]\!]_C^* \subseteq [\![A^*]\!]_C$, the converse inclusion cannot be established (this is a standard fact about PDL, see [7]). C is similar to dynamic models in the other aspects.

$F \subseteq \mathsf{Form}$ is *Fisher–Ladner closed* iff it is closed under subformulas and

- if $[X?]Y \in F$, then $X \in F$
- if $[A \cup B]X \in F$, then $[A]X \in F$ and $[B]X \in F$
- if $[A; B]X \in F$, then $[A][B]X \in F$
- if $[A^*]X \in F$, then $[A][A^*]X \in F$
- variants of the above with $\langle\,\rangle$

The *Fisher–Ladner closure* $FL(F)$ of F is the least Fisher–Ladner closed superset of F; the Fisher-Ladner closure of $X \vdash Y$ is $FL(\{X, Y\})$. If F is finite, then $FL(F)$ is finite.

Let $F = FL(F)$. We define $\equiv^F$ on P by $t \equiv^F t' \iff t \cap F = t' \cap F$. Let t^F be the $\equiv^F$-equivalence class containing t. The *filtration of C through F* is $M^F = \langle W^F, [\![\,]\!]^F\rangle$ where

- $W^F = \{t^F\ ;\ t \in P\}$
- $[\![p]\!]^F = \{t^F\ ;\ p \in t\}$ for $p \in F$; for $p \notin F$ we set $[\![p]\!]^F = \emptyset$
- $t^F[\![a]\!]^F s^F$ iff there are $t' \in t^F$ and $s' \in s^F$ such that $t'[\![a]\!]s'$

$[\![A]\!]^F$ and $[\![X]\!]^F$ for non-atomic A, X are defined as in dynamic models. Hence, M^F is a dynamic model by definition. We write $t[\![A]\!]^F s$ instead of $t^F[\![A]\!]^F s^F$ and $t \in [\![X]\!]^F$ or $t \vDash^F X$ instead of $t^F \in [\![X]\!]^F$.

Theorem 2.1 (Filtration Theorem) *If $F = FL(F)$, then $t \vDash X$ iff $t \vDash^F X$ for all $X \in F$.*

Proof. (Sketch) The crucial part of the proof is to show that if $t[\![A^*]\!]s$ in the canonical structure, then $t[\![A^*]\!]^F s$ in the filtration, i.e. that there is a finite sequence of t_i^F such that $t_0^F = t^F$ and $t_n^F = s^F$ and $t_k[\![A]\!]^F t_{k+1}$ for all $k < n$. In the case of full PDL, an important lemma on the way to showing this is that the set of u such that u^F is reachable from t^F in a finite number of $[\![A]\!]^F$-

steps is definable by a particular formula Z (in the sense that u is in the set iff $Z \in u$); in the context of full PDL, the definition of this Z, however, relies heavily on Boolean negation.

In the context of positive PDL we solve this by devising a formula that defines the *complement* of each set of non-empty prime theories (satisfying certain assumptions). The crucial aspect of the positive language that allows us to do this is the presence of the test operator as test formulas simulate material implication. $\square$

Theorem 2.2 *$X \vdash Y$ is in PDL^+ iff it is provable in $\mathcal{PDL}^+$.*

3 Non-classical extensions of positive PDL

The main application of our results on PDL^+ is in the context of completeness proofs of non-classical extensions of PDL^+. The only thing that remains to be shown in the context of such extensions, usually defined in terms of a class of models, is that the filtration of the canonical structure belongs to the class of models. Two examples of this will be briefly outlined, namely, a version of PDL with a De Morgan negation and an extension of PDL with the residual operations of the Lambek calculus. Below we summarize the main definitions and results.

3.1 Non-classical extensions I: Adding De Morgan negation

The language $\mathcal{L}^\sim$ extends $\mathcal{L}$ with $\sim$. A *dynamic De Morgan model* is $M = \langle W, \sim, [\![\,]\!]\rangle$ where $\langle W, [\![\,]\!]\rangle$ is a dynamic model, $w^{\sim\sim} = w$ and $[\![\sim X]\!] = \{w \,;\, w^\sim \not\models X\}$; $PDL^\sim$ is the set of all consequence pairs valid in all dynamic De Morgan models. The proof system $\mathcal{PDL}^\sim$ is $\mathcal{PDL}^+$ extended with

$$X \dashv\vdash {\sim\sim}X \qquad {\sim}X \wedge {\sim}Y \vdash {\sim}(X \vee Y) \qquad {\sim}(X \wedge Y) \vdash {\sim}X \vee {\sim}Y \qquad \frac{X \vdash Y}{{\sim}Y \vdash {\sim}X}$$

Theorem 3.1 *$X \vdash Y$ belongs to $PDL^\sim$ iff it is provable in $\mathcal{PDL}^\sim$.*

Proof. In the context of $PDL^\sim$, we need to re-define the notion of a Fisher–Ladner closed set; we require that if $X \in F$ and $X \neq {\sim}Y$ (for all $Y \in \mathcal{L}^\sim$), then ${\sim}X \in F$. It follows that $t \equiv^F u$ implies $t^\sim \equiv^F u^\sim$ and, therefore, the filtration of the canonical structure is well defined. The proof for PDL^+ takes care of the rest. (The canonical structure C consists of prime theories that are not the full language and $t^{\sim_C}$ is defined as usual— $t^{\sim_C} = \{X \,;\, {\sim}X \notin t\}$) $\square$

Theorem 3.2 *$PDL^\sim$ is decidable.*

We also established decidability and completeness of a special version of $PDL^\sim$ suitable for reasoning about additions of positive and negative information to so called Belnapian databases [1], but we omit details because of space limitations.

3.2 Non-classical extensions II: Lambek PDL

The dynamic Lambek language $\mathcal{L}^\backslash$ adds binary $\backslash, \cdot, /$ to $\mathcal{L}$. A *dynamic Lambek model* is $M = \langle W, R, [\![\;]\!]_M\rangle$ where $\langle W, [\![\;]\!]_M\rangle$ is a dynamic model and R is a ternary relation on W. It is assumed that $[\![X{\cdot}Y]\!]_M$, $[\![X\backslash Y]\!]_M$ and $[\![X/Y]\!]_M$

are defined as usual [3,9]. $PDL^{\backslash}$ is the set of consequence $\mathcal{L}^{\backslash}$-pairs valid in all dynamic Lambek models. $\mathcal{PDL}^{\backslash}$ is $\mathcal{PDL}^{+}$ extended with

Axioms $X{\cdot}(X\backslash Y) \vdash Y \quad (Y/X){\cdot}X \vdash Y \quad Y \vdash X\backslash(X{\cdot}Y) \quad Y \vdash (Y{\cdot}X)/X$

Rules $\dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{X_1{\cdot}X_2 \vdash Y_1{\cdot}Y_2} \qquad \dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{Y_1\backslash X_2 \vdash X_1\backslash Y_2} \qquad \dfrac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2}{X_2/Y_1 \vdash Y_2/X_1}$

Theorem 3.3 *$X \vdash Y$ belongs to $PDL^{\backslash}$ iff it is provable in $\mathcal{PDL}^{\backslash}$.*

Proof. The Truth Lemma for the Lambek connectives (proven as in [12]) is independent of the rest, established already within the proof for PDL^{+}. □

Theorem 3.4 *$PDL^{\backslash}$ is decidable.*

We obtained similar results for extensions of $PDL^{\backslash}$ with commutativity $X{\cdot}Y \vdash Y{\cdot}X$ and weak contraction $X \vdash X{\cdot}X$.

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Logical Connectives for some FDE-based Modal Logics

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Abstract

Model-theoretic proofs of the definability of connectives along the lines of [1] and [5] are given for various FDE-based modal logics.

Keywords: FDE, Modal Logic, Functional Completeness, Definable Connectives

1 Introduction

This note studies functional completeness or, better, the class of definable connectives for some FDE-based modal logics. The results of this note are along the lines of [1] and [5], where results about the class of definable connectives were given with respect to intuitionistic logic and various constructive modal logics with strong negation.

For many-valued logics the notion of functional-completeness is well understood. Given a many-valued logic and a finite set of n-ary truth-function, if every possible n-ary truth-function of the logic in question can be obtained by finite compositions of the given set of truth-functions, this set is called functionally complete. In particular, the search space for the functional completeness result is well defined, namely the class of all possible n-ary truth-functions. For relational semantics as for FDE-based modal logics, however, it is not clear what the search space should be, since the truth and falsity conditions are given in terms of metalogical verification and falsification clauses. Our strategy for obtaining results about the definable connectives is therefore somewhat different from that for many-valued logic. We will first restrict the class of expressible metalogical verification and falsification conditions, and then show that within this restricted class all metalogically expressible truth and falsity conditions can be expressed by object language formulas of the respective logics.

This note is organized as follows. In section 2 we will recall the semantical definitions of the systems **KFDE**, **KN4**, $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$, **BK**. In section 3 we will explore the classes of definable logical connectives in the various modal extensions of **FDE** and section 4 contains some remarks on the restrictions of the presented method and briefly mentions a possible alternative proof strategy for functional completeness of FDE-based modal logics.

2 FDE-based modal logics

The languages $\mathcal{L}_{\mathbf{KFDE}} = \{\vee, \wedge, \sim, \Box\}$, $\mathcal{L}_{\mathbf{KN4}} = \{\vee, \wedge, \Rightarrow, \sim, \Box\}$, $\mathcal{L}_{\mathbf{BK}^{\Box -}} = \{\vee, \wedge, \rightarrow, \sim, \Box\}$, $\mathcal{L}_{\mathbf{BK}^{\Box}} = \{\vee, \wedge, \rightarrow, \sim, \Box, \bot\}$ and $\mathcal{L}_{\mathbf{BK}} = \{\vee, \wedge, \rightarrow, \sim, \Box, \Diamond, \bot\}$ are based on a non-empty countable set of atomic propositions Prop. We denote by $\mathsf{Form}(\mathsf{L})$, where L stands for the respective logic, the set of formulas defined as usual, formulas by A, B, C, etc. and sets of formulas by Γ, Δ, Σ, etc.

An L-model is a tuple $\mathcal{M} = \langle W, R, v^+, v^- \rangle$, where W is a non-empty set of information states (possible worlds), $R \subseteq W^2$ is an accessibility relation on W, and v^+ and v^- are functions $v^+, v^- : \mathsf{Prop} \rightarrow 2^W$. We now define verification and falsification relations $\Vdash^+$ and $\Vdash^-$ between worlds and formulas in model $\mathcal{M}$ as follows [1]:

$$
\begin{array}{lll@{\qquad}lll}
w \Vdash^+ p & \text{iff} & w \in v^+(p); & w \Vdash^+ A \Rightarrow B & \text{iff} & ((w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B) \\
w \Vdash^- p & \text{iff} & w \in v^-(p); & & & \Wedge (w \Vdash^- B \twoheadrightarrow w \Vdash^- A)); \\
w \Vdash^+ A \wedge B & \text{iff} & (w \Vdash^+ A \Wedge w \Vdash^+ B); & w \Vdash^- A \Rightarrow B & \text{iff} & (w \Vdash^+ A \Wedge w \Vdash^- B); \\
w \Vdash^- A \wedge B & \text{iff} & (w \Vdash^- A \Vee w \Vdash^- B); & w \Vdash^+ A \rightarrow B & \text{iff} & (w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B); \\
w \Vdash^+ A \vee B & \text{iff} & (w \Vdash^+ A \Vee w \Vdash^+ B); & w \Vdash^- A \rightarrow B & \text{iff} & (w \Vdash^+ A \Wedge w \Vdash^- B); \\
w \Vdash^- A \vee B & \text{iff} & (w \Vdash^- A \Wedge w \Vdash^- B); & w \Vdash^+ \Box A & \text{iff} & \forall u (wRu \twoheadrightarrow u \Vdash^+ A); \\
w \Vdash^+ {\sim} A & \text{iff} & w \Vdash^- A; & w \Vdash^- \Box A & \text{iff} & \exists u (wRu \Wedge u \Vdash^- A); \\
w \Vdash^- {\sim} A & \text{iff} & w \Vdash^+ A; & w \Vdash^+ \Diamond A & \text{iff} & \exists u (wRu \Wedge u \Vdash^+ A); \\
\neg (w \Vdash^+ \bot) & \text{and} & w \Vdash^- \bot; & w \Vdash^- \Diamond A & \text{iff} & \forall u (wRu \twoheadrightarrow u \Vdash^- A);
\end{array}
$$

We say a formula A is true at world w in an L-model $\mathcal{M}$, iff $w \Vdash^+ A$. We say a formula A is true in an L-model, $\mathcal{M} \Vdash^+ A$, iff A is true at every world w in $\mathcal{M}$. A formula A is L-valid, $\vDash_{\mathsf{L}} A$, iff A is true in every L-model. Finally, a set of formulas Γ entails a formula A, $\Gamma \vDash_{\mathsf{L}} A$, iff for all L-models $\mathcal{M}$ and worlds w, if $w \Vdash^+ B$, for all $B \in \Gamma$, then $w \Vdash^+ A$.

3 Logical Connectives for some FDE-based Modal Logics

In this section we will follow and extend the proofs of [5], where itself the results from [1] were extended. Note that at first we will use almost the same definitions as in [5]. We will make this clear by referring to the original definitions and proofs.

The results presented here should not be understood as results of functional completeness in the usual model theoretical sense, even though we will use this term, but as results about the definable class of connectives in the various modal extensions of **FDE**.

We begin by defining the vocabulary of the metalogical formulas for the various modal extensions of **FDE**, simultaneously. The metalogical language is a two-sorted first-order language containing all formulas of $\mathsf{Form}(\mathsf{L})$ as the first sort of individual variables, a non-empty denumerable set V of information state variables as the second sort of variables, the classical connectives $\Wedge$, $\Vee$,

---

[1] We use a classical metalanguage for defining the verification and falsification relations.

$\neg$, $\twoheadrightarrow$, the classical quantifiers $\forall$ and $\exists$ and the binary predicate symbols $\Vdash^+$, $\Vdash^-$ and R. The metalanguage is then defined as follows:

state variables:	$w \in V$
object language formula variables:	$A \in \mathsf{Form}(\mathsf{L})$
atomic formulas of the metalanguage:	α
formulas of the metalanguage:	φ

$\alpha ::= w \Vdash^+ A \mid w \Vdash^- A \mid wRw$

$\varphi ::= \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \twoheadrightarrow \varphi \mid \forall \varphi \mid \exists \varphi$

Bi-implication $\leftrightarrow$ is defined as usual.

If a formula φ contains the free variables $w, A_1, \ldots, A_n$, we will write this as $\varphi(w, A_1, \ldots, A_n)$. Formulas wRu are called relational atoms. In formulas $\forall u(wRu \twoheadrightarrow \varphi(u))$ and $\exists u(wRu \wedge \varphi(u))$ the relational atom wRu is said *to occur as a lower bound* on the quantifier $\forall u$ ($\exists u$). And the quantifier $\forall u$ ($\exists u$) is said *to be bounded below* by wRu. Note that in such formulas, u is the only free variable.

Definition 3.1 [Definition 1 in [5, p. 471]] A formula φ is a *regular* metalogical formula iff

(i) φ contains at most one free state variable,

(ii) all relational atoms occur as a lower bound on a quantifier,

(iii) every quantifier is bounded below by a relational atom.

Remark 3.2 Every regular metalogical formula φ is of such shape that every quantified subformula of φ has the shape $\forall u(wRu \twoheadrightarrow \theta)$ or $\exists u(wRu \wedge \theta)$, where θ has no free state variables other than u.

Definition 3.3 [Definition 2 in [5, p. 471]] Let φ be a regular metalogical formula. The formula $\overline{\varphi}$ is inductively defined as follows:

$$\begin{array}{rcl} \overline{\alpha} &=& \alpha,\ \alpha \text{ is an atom} \\ \overline{\neg wRu} &=& \neg wRu \\ \overline{\neg w \Vdash^+ A} &=& w \Vdash^- A \\ \overline{\neg w \Vdash^- A} &=& w \Vdash^+ A \\ \overline{\neg\neg\theta} &=& \overline{\theta} \\ \overline{\psi \circ \theta} &=& \overline{\psi} \circ \overline{\theta}, \circ \in \{\wedge, \vee, \twoheadrightarrow\} \\ \overline{\neg(\psi \wedge \theta)} &=& \overline{\neg\psi} \vee \overline{\neg\theta} \end{array} \qquad \begin{array}{rcl} \overline{\neg(\psi \vee \theta)} &=& \overline{\neg\psi} \wedge \overline{\neg\theta} \\ \overline{\neg(\psi \twoheadrightarrow \theta)} &=& \overline{\psi} \wedge \overline{\neg\theta} \\ \overline{\forall w\psi} &=& \forall w\overline{\psi} \\ \overline{\exists w\psi} &=& \exists w\overline{\psi} \\ \overline{\neg\forall w\psi} &=& \exists w\overline{\neg\psi} \\ \overline{\neg\exists w\psi} &=& \forall w\overline{\neg\psi} \end{array}$$

If φ is a regular metalogical formula, then $\overline{\varphi}$ is said to be an L-*regular* metalogical formula.

Remark 3.4 Note that the notion of **KFDE**-regular connectives requires an additional condition for $\twoheadrightarrow$:

$$\overline{\neg(\psi \twoheadrightarrow \theta)} = \overline{\psi} \wedge \overline{\neg\theta}$$

This additional condition restricts the set of **KFDE**-regular connectives since a metalogical implication can not be expressed in the object language of **KFDE**.

Definition 3.5 [Definition 3 in [5, p. 472]] A definition of an n-ary ($n \geq 1$) connective $\bigstar$ has the form:

For every model $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ and every $w \in W$,

$w \Vdash^+ \bigstar(A_1, \ldots, A_n)$ iff $\overline{\psi}(w, (A_1, \ldots, A_n))$ and $w \Vdash^- \bigstar(A_1, \ldots, A_n)$ iff $\overline{\neg\psi}(w, (A_1, \ldots, A_n))$, where $\overline{\psi}$ is an L-regular metalogical formula, by which $\bigstar$ is said to be defined.

Definition 3.6 [Definition 5 [5, p. 472]] The *degree of quantification* $d(\varphi)$ of a metalogical formula φ is inductively defined as follows:

$d(\varphi) = 0$, if φ is quantifier-free;

$d(\varphi) = \max(n_i + 1)$ for $1 \leq i \leq j$, if φ has j quantifiers, the i-th quantifier ranges over the subformula δ_i of φ, and n_i is the degree of quantification of δ_i

Remark 3.7 Every L-regular metalogical formula φ is of such shape that every quantified subformula of φ has the form $\forall u(wRu \twoheadrightarrow \theta)$ or $\exists u(wRu \mathbin{\&} \theta)$, where θ has no free state variable other than u.

Remark 3.8 [cf. Corollary 1 in [5, p. 474]] Let φ be an L-regular metalogical formula. For every subformula of φ of the shape $\forall u(wRu \twoheadrightarrow \theta)$ or $\exists u(wRu \mathbin{\&} \theta)$, θ is L-regular.

Remark 3.9 For each L-formula A, $\overline{\varphi}$ iff $w \Vdash^+ A$ just in case $\overline{\neg\varphi}$ iff $w \Vdash^- A$.

Definition 3.10 A logical connective that can be defined by means of an L-regular metalogical formula is said to an L-*regular connective.*

Remark 3.11 The sets of connectives $\{\vee, \wedge, \sim, \Box\}$, $\{\vee, \wedge, \sim, \Rightarrow, \Box\}$, $\{\vee, \wedge, \sim, \rightarrow, \Box\}$, $\{\vee, \wedge, \sim, \rightarrow, \bot, \Box\}$ and $\{\vee, \wedge, \sim, \rightarrow, \bot, \Box, \Diamond\}$ are **KFDE**-, **KN4**-, $\mathbf{BK}^{\Box-}$–, $\mathbf{BK}^{\Box}$- and **BK**-regular, respectively.

Theorem 3.12 *In the class of* L-*regular connectives the respective sets of connectives are functionally complete. I.e., if an n-ary ($n \geq 1$) connective $\bigstar$ is defined by means of an* L-*regular metalogical formula $\overline{\varphi}$, then there is an* L-*formula A such that the following holds: $\overline{\varphi} \leftrightarrow w \Vdash^+ A$ (and $\overline{\neg\varphi} \leftrightarrow w \Vdash^- A$).*

Proof. By induction on the degree of quantification of $\overline{\varphi}$.

Suppose $d(\overline{\varphi}) = 0$, then every atomic subformula of $\overline{\varphi}$ is a non-relational atom and every metalogical operator occurring in $\overline{\varphi}$ is either $\mathbin{\&}$, $\mathbin{\mathrm{V}}$ or $\twoheadrightarrow$. Let A be the result of replacing every occurrence of $\mathbin{\&}$ by $\wedge$, every occurrence of $\mathbin{\mathrm{V}}$ by $\vee$, every occurrence of $w \Vdash^+ B$ by B and every occurrence of $w \Vdash^- B$ by $\sim B$. In case of $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$ and **BK** we furthermore replace every occurence of $w \Vdash^{\pm} A \twoheadrightarrow w \Vdash^{\pm} B$ by $A^* \rightarrow B^*$ and in case of **KN4** every occurence of $w \Vdash^{\pm} A \twoheadrightarrow w \Vdash^{\pm} B$ by $(A^* \Rightarrow (A^* \Rightarrow B^*)) \vee B^*$, where A^* and B^* are the respective replacements of $w \Vdash^{\pm} A$ and $w \Vdash^{\pm} B$. In case of $\mathbf{BK}^{\Box}$ and **BK** we furthermore replace every occurence of $w \Vdash^+ \bot$ by $\sim\bot$ and $w \Vdash^- \bot$ by $\bot$. Then $\overline{\varphi} \leftrightarrow w \Vdash^+ A$ (as well as $\overline{\neg\varphi} \leftrightarrow w \Vdash^- A$), cf. Observation 3.9.

Now, let $d(\overline{\varphi}) > 0$. Then there is a subformula of $\overline{\varphi}$ of the shape $\forall u(wRu \twoheadrightarrow \theta)$ or $\exists u(wRu \mathbin{\&} \theta)$, where θ is quantifier free. By the induction base case $\theta \leftrightarrow A$ for some L-formula. Now we have $\forall u(wRu \twoheadrightarrow \theta)$ iff $w \Vdash^+ \Box A$ and $\exists u(wRu \mathbin{\&} \theta)$ iff $w \Vdash^+ \sim\Box\sim A$. Moreover, $\exists u(wRu \mathbin{\&} \theta)$ iff $w \Vdash^+ \Diamond A$.

If the subformulas $\forall\!\!\!\forall\, u(wRu \twoheadrightarrow \theta)$ and $\exists\!\!\!\exists\, u(wRu \mathbin{\wedge\!\!\!\wedge} \theta)$ in $\overline{\varphi}$ are replaced by their respective equivalents the result is an L-regular metalogical formula which has one less quantifier than $\overline{\varphi}$, and hence the induction hypothesis can be used. □

4 Short Discussion of the Results

Limits of the presented method

The results presented above are in need of some clarification. At first glance Theorem 3.12 seems to state the rather trivial result that everything that is expressible is expressible. However, the result not only shows what *is* expressible, but what is *not*, as well. The proof for functional completeness heavily relies 1) on the notion of L-regular connectives and 2) on the requirement that verification and falsification can be expressed by one metalogical formula as in Observation 3.9. As for 1), in the class of L-regular connectives it is, for example, not possible to distinguish between $\neg w \Vdash^+ A$ and $w \Vdash^- A$ ($\neg w \Vdash^- A$ and $w \Vdash^+ A$). This means that operators like $\circlearrowright$ which make **FDE** functionally complete, cf. [4], with the following verification and falsification conditions: $w \Vdash^+ \circlearrowright A$ iff $(w \Vdash^+ A \mathbin{\wedge\!\!\!\wedge} w \neg \Vdash^- A) \mathbin{\vee\!\!\!\vee} (\neg w \Vdash^+ A \mathbin{\wedge\!\!\!\wedge} w \neg \Vdash^- A)$ and $w \Vdash^- \circlearrowright A$ iff $(w \Vdash^+ A \mathbin{\wedge\!\!\!\wedge} w \neg \Vdash^- A) \mathbin{\vee\!\!\!\vee} (w \Vdash^+ A \mathbin{\wedge\!\!\!\wedge} w \Vdash^- A)$ can not be expressed by means of the given languages, as it should be. As for 2), the requirement that verification and falsification need to be expressed by one metalogical formula, limits the class of languages to which the presented method can be applied to. Similar things can be said about the results in [5]. Therefore, in languages that contain for example $\circlearrowright$ or $\otimes$ and $\oplus$, cf. [3], where truth and falsity conditions can be seen as *asymmetric*, we need an alternative method to obtain results about the classes of definable connectives.

Future Work

In [2] and [3] the semantics for the FDE-based modal logics is given in terms of so-called twist-structures, where *truth-values* are structured two-dimensional objects, with classical truth-functions for the **FDE** operators, operating independently in every dimension. Since in such setting one can use methods from many-valued logic, in the future we will investigate the notion of definable connectives for FDE-based modal logics and functional completeness for twist-structure semantics.

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Active structural completeness for tabular modal logics

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The notions of admissibility and structural completeness have received considerable attention for many years. Let us start with the definition of admissibility for normal modal logics. Let L be such a logic and r be a rule. Then r is *admissible* for L if there is no proof for any formula outside L with the use of the modus ponens rule, the necessitation rule, axioms of L, and r. A logic L is structurally complete (SC) if all of its admissible rules are derivable from the modus ponens rule, the necessitation rule and axioms of L.

For some important systems, like the modal logic S5, failure of structural completeness is caused only by the underivability of a passive rule. A rule $r = \Gamma/\gamma$ is *active* for L if there is a substitusion σ such that $\sigma(\Gamma) \subseteq L$. Otherwise, r is called *passive* for L. Neglecting passive rules in the definition of SC led Dzik in [5] to introduce a new property of *active structural completeness* (ASC) (called there almost structural completeness), see also [6].

We would like to check whether this slight change in the definition is essential. Both properties reflect the same idea. Indeed, passive rules cannot be used in any proof which starts with axioms of L. But, at least for modal logics, SC is rare.

The result is that among normal modal logics given by frames with less than 7 vertices (there are 96709517 such logics) we have around 5 % of SC and 76 % of ASC logics. Among normal transitive modal logics given by frames with less than 9 vertices (there are 2091055 such logics) we have around 2 ‰ of SC and 1 % of ASC logics. Perhaps the most interesting observation from our experiment is that the ratio of ASC logics is quite stable when the frame size in growing (at least to the level we managed to check). It is not the case for SC.

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Also, it is not the case for both properties when we restrict to the transitive case, see tables.

The results are obtained by appropriate modifications of the known algorithms which checks both properties for finitely generated quasivarieties of algebras [10].

1 Basic algorithm

It appeared that the SC and ASC properties are decidable for finitely generated quasivarieties of arbitrary algebras. The first algorithm for SC was given by Dywan in [4] (an input for the algorithm is a family of algebras generating a quasivariety of interest). Another algorithm for SC may be deduced from the work of Bergman [1]. An appropriate modification of Dywan's algorithm allows checking ASC too. A modification of Bergman's algorithm for ASC is more complicated. It was presented by Metcalfe and Röthlisberger in [10]. We use this algorithm.

It is an open problem whether the SC and ASC properties are decidable for finitely generated varieties. However, the recalled results allow deciding (A)SC for finitely generated congruence distributive varieties. Indeed, by Jónsson's Lemma [7], such varieties are finitely generated also as quasivarieties, and we can easily find the generators.

Although, the above algorithms are rather algebraic in nature, for modal logics we may dualize them to relational semantics. The profit of expressing the algorithms in relational language is the logarithmic reduction of the size of considered structures. A slightly modified and dualized algorithm of Metcalfe and Röthlisberger for serial, i.e., satisfying $\Diamond\top$, modal frames is presented below. (One could also derive a version of Metcalfe's and Röthlisberger's algorithm for arbitrary frames. However, we will show an easier way to deal with non-serial frames.)

Input: A finite frame $\mathbf{W}$
Output: Is $L(\mathbf{W})$ ASC?
  $\mathcal{R} \leftarrow$ the set of rooted generated subframes of $\mathbf{W}$
  for $\mathbf{R} \in \mathcal{R}$ **do**
    $r_{\mathbf{R}} \leftarrow$ minimal number k of variables for which there exists a valuation v with k variables s.t. $(\mathbf{R}, v)$ does not have nontrivial bisimulation
  end for
  $r \leftarrow \max\{r_{\mathbf{R}} \mid \mathbf{R} \in \mathcal{R}\}$
  $\mathbf{U} \leftarrow$ a universal frame for $L(\mathbf{W})$ of rank r
  for $\mathbf{R} \in \mathcal{R}$ **do**
    if $\mathbf{R}$ and $\mathbf{R} + \bullet$ are not p-morphic images of $\mathbf{U}$ **then** Return No
    end if
  end for
  Return Yes

Algorithm 1. Basic algorithm checking ASC for frames satisfying $\Diamond\top$.

Explanations to Algorithm 1: $L(\mathbf{W})$ is the logic characterized by a frame $\mathbf{W}$; A universal frame $\mathbf{U}$ of rank r is a dual to a free algebra of rank r (for the variety generated by an algebra which is dual to $\mathbf{W}$), see e.g. [3]. $\bullet$ denotes a frame consisting of one reflexive vertex, and $+$ denotes a disjoint union of frames.

This basic algorithm allows checking ASC for most of 5-element frames. However, it is too slow for larger frames. The main obstacle is the size of a universal frame $\mathbf{U}$. Indeed, though the number r is small (it is bounded by $\log |W|$), the cardinality of the carrier of $\mathbf{U}$, in general, depends exponentially on $|W|$ (it is bounded by $|W| \cdot 2^{|W| \log |W|}$). The algorithm runs in 2EXPTIME. The reasoning behind the algorithms shows that checking (A)SC is in NEXPTIME.

2 Improvements

Our changes in the basic algorithm are mainly of two sorts. Firstly, we did not use basic algorithms when we can apply some known and some new conditions which yield (A)SC or yield non-(A)SC.

For example, if a logic L has projective unification, then it is ASC [5]. We have a computationally easy characterization of normal extensions of K4 with projective unification [8]. Moreover, if the reflexive transitive closure of a finite frame $\mathbf{W}$ consists of a disjoint union of frames with total relations, then the variety of modal algebras which is the algebraic counterpart of $L(\mathbf{W})$ is a discriminator variety [9]. This property yields projective unification [2], and hence ASC. Note that all finite symmetric frames fall into this class.

As an example of a new fact used to decide ASC, let us present the following statement. It allows us to reduce decidability of ASC for arbitrary frames to the case of serial frames.

Proposition 2.1 *Let $\mathbf{W}$ be a finite frame. Then $L(\mathbf{W})$ is ASC if and only if $\mathbf{W}$ is a disjoint union of a serial frame $\mathbf{W_s}$ such that $L(\mathbf{W_s})$ is ASC and a frame with the empty accessibility relation.*

Secondly, it is possible to replace a universal frame in the basic algorithm for a family of smaller objects. This idea goes back to [10]. Indeed, as shown there, we may take a family $\mathcal{U}$ of p-morphic images of a universal frame $\mathbf{U}$ such that every rooted generated subframe of $\mathbf{W}$ is a generated subframe of one frame from $\mathcal{U}$. Instead of getting into details on the construction of $\mathcal{U}$, let us only indicate that we may find it without computing $\mathbf{U}$. Let $\mathcal{R}$ be a family of maximal rooted subframes of $\mathbf{W}$ and let $\mathbf{M}_i = (\mathbf{R}_i, v_i)$, $i < m$, be a list of all models such that $\mathbf{R}_i \in \mathcal{R}$ and the valuations v_i have variables in $\{p_1, \ldots, p_r\}$ (r is the number computed in the algotithm). Then $\mathbf{U} \cong \sum_i \mathbf{R}_i/\alpha$, where α is a largest bisimulation of $\sum_i \mathbf{M}_i$. Let $\mathbf{R} \in \mathcal{R}$. Let us suppose that we want to compute $\mathbf{U_R}$ such that $\mathbf{U_R}$ is a p-morphic image of $\mathbf{U}$ and $\mathbf{U_R}$ has a generated subframe isomorphic to $\mathbf{R}$. Assume that $\mathbf{R} \cong \mathbf{R}_0$. Then for each $0 \leq j \leq m$ we find a frame, as small as we can, of the form $\mathbf{U}_j = \sum_{i=0}^{j} \mathbf{R}_i/\beta_j$, where β_j are bisimilar equivalence relations, β_0 is the identity, $\beta_j = \beta_{j+1} \cap (\sum_{i=0}^{j} R_i)^2$,

Fig. 1. Some 6-element frames which we failed to check.

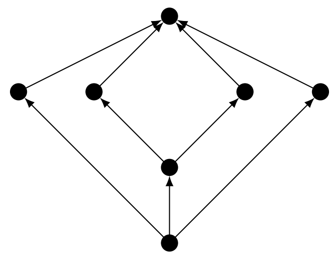

Fig. 2. The 7-element transitive frame which we failed to check.

and $\alpha \cap (\sum_{i=0}^{j} R_i)^2 \subseteq \beta_j$. At j'th step of computation we consider a certain model with $\mathbf{U}_{j-1}$ as the frame reduct and a model $\mathbf{M}_j$. This allows us to work all the time on relatively small objects.

3 Results

The following tables summarize results of our computations. Note that we count logics, not frames (ASC and SC are properties of logics, not of frames).

column	definition
size	cardinality of the carrier of a frame
logics	nr of logics of any frame of a given size but not of smaller size
UN	(unchecked) nr of logics for which we failed to check ASC and SC
SC	% of SC logics among counted logics
ASC	% of ASC logics among counted logics

Table 1
Legend for tables

size	logics	UN	SC	ASC
Normal extensions of K				
1	2	0	100	100
2	8	0	25.0	75.0
3	86	0	14.6	62.9
4	2838	0	9.4	60.1
5	285799	0	6.8	66.0
6	96420781	14	5.2	76.0
Normal extensions of D=K$\oplus\Diamond\top$				
1	1	0	100	100
2	5	0	40.0	100
3	62	0	24.2	85.5
4	2214	0	12.1	74.4
5	244134	0	7.9	76.4
6	87722854	14	5.7	83.3
Normal extensions of T=K$\oplus\Box p \to p$				
1	1	0	100	100
2	2	0	50.0	100
3	12	0	41.7	100
4	189	0	23.8	83.1
5	9175	0	15.6	79.3
6	1523497	1	11.0	83.9

Table 2
Normal extensions of K

size	logics	UN	SC	ASC
Normal extensions of K4=K$\oplus\Box p \to \Box\Box p$				
1	2	0	100	100
2	6	0	33.3	66.6
3	26	0	19.2	42.3
4	145	0	13.8	27.6
5	1050	0	6.4	12.6
6	9917	0	2.3	4.6
7	121496	1	0.7	1.5
8	1958413	1.8 ‰	0.2	1.2
Normal extensions of KD4=K4$\oplus\Diamond\top$				
1	1	0	100	100
2	3	0	66.6	100
3	11	0	63.6	90.9
4	52	0	38.5	57.7
5	315	0	21.2	32.4
6	2496	0	9.3	14.0
7	26314	1	3.0	4.6
8	370304	0.9 %	0.8	1.2
Normal extensions of S4=K4$\oplus\Box p \to p$				
1	1	0	100	100
2	2	0	50	100
3	5	0	60	100
4	15	0	40	73.3
5	55	0	32.7	56.4
6	242	0	19.4	33.9
7	1322	1	10.5	18.0
8	9160	5.4 %	4.6	7.8

Table 3
Normal extensions of K4

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Zero-one laws with respect to models of provability logic and two Grzegorczyk logics

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1 Introduction

In the late 1960s, Glebskii and colleagues proved that first-order logic without function symbols satisfies a zero-one law: every formula is either almost always true or almost always false in finite models [6]. More formally, let L be a language of first-order logic and let $A_n(L)$ be the set of all L-models with universe $\{1, \ldots, n\}$. Now let $\mu_n(\sigma)$ be the fraction of members of $A_n(L)$ in which σ is true:

$$\mu_n(\sigma) = \frac{\mid M \in A_n(L) : M \models \sigma \mid}{\mid A_n(L) \mid}$$

Then for every $\sigma \in L$, $\lim_{n\to\infty} \mu_n(\sigma) = 1$ or $\lim_{n\to\infty} \mu_n(\varphi) = 0$. This was also proved later but independently by Fagin [5]; Carnap had already proved the zero-one law for first-order languages with only unary predicate symbols [3].

The above zero-one laws and other limit laws have found applications in database theory and AI.In this article, we are interested in zero-one laws for some modal logics that impose structural restrictions on their models; all three logics that we are interested in are sound and complete with respect to finite partial orders, with different extra restrictions per logic. The zero-one law for first-order logic also holds when restricted to partial orders, both reflexive and irreflexive ones [4]. The proof uses a surprising combinatorial result by Kleitman and Rothschild [9] on which we will also rely for our results.

1.1 Kleitman and Rothschild's result on finite partial orders

Kleitman and Rothschild proved that with asymptotic probability 1, finite partial orders have a very special structure: There are no chains $u < v < w < z$ of more than three objects and the structure can be divided into three levels:

- L_1, the set of minimal elements;
- L_2, the set of elements immediately succeeding elements in L_1;
- L_3, the set of elements immediately succeeding elements in L_2.

Moreover, in partial orders of size n, the sizes of these sets tend to $\frac{n}{4}$ for both L_1 and L_3 while the size of L_2 tends to $\frac{n}{2}$. As n increases, each element

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in L_1 has as immediate successors asymptotically half of the elements of L_2 and each element in L_2 has as immediate successors asymptotically half of the elements of L_3 [9]. Kleitman and Rothschild's theorem holds both for reflexive (non-strict) and for irreflexive (strict) partial orders.

1.2 Zero-one laws for modal logics

Let $\Phi = \{p_1, \ldots, p_k\}$ be a finite set of propositional atoms [2] and let $L(\Phi)$ be the modal language over Φ, inductively defined as the smallest set closed under:

(i) If $p \in \Phi$, then $p \in L(\Phi)$.

(ii) If $A \in L(\Phi)$ and $B \in L(\Phi)$, then also $\neg A \in L(\Phi)$, $\Box A \in L(\Phi)$, $\Diamond(\varphi) \in L(\Phi)$, $(A \wedge B) \in L(\Phi)$, $(A \vee B) \in L(\Phi)$, and $(A \to B) \in L(\Phi)$.

Let $\mathcal{M}_{n,\Phi}$ be the set of finite Kripke models over Φ with set of worlds $W = \{1, \ldots, n\}$. We take $\nu_{n,\Phi}$ to be the uniform probability distribution on $\mathcal{M}_{n,\Phi}$. Let $\nu_{n,\Phi}(\varphi)$ be the measure in $\mathcal{M}_{n,\Phi}$ of the set of Kripke models in which φ is valid. Halpern and Kapron proved that every formula φ in $L(\Phi)$ is either valid in almost all models or not valid in almost all models [8, Corollary 4.2]:

$$\text{Either } \lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 0 \text{ or } \lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1.$$

By the Kleitman-Rothschild theorem, this modal zero-one law can also be restricted to finite models on reflexive or irreflexive partial orders, so that the existence of zero-one laws for finite models of provability logic and Grzegorczyk logic immediately follow. However, one would like to prove a stronger result and axiomatize the set formulas φ for which $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$.

The result about **GL** was proved in my 1995 LMPS presentation [12], but the proof was not published before. The 0-1 laws for **Grz** and **WGrz** are new.

2 Provability logic and two of its cousins

Here follow brief reminders about provability logic **GL**, Grzegorczyk logic **Grz**, and weak Grzegorczyk logic **wGrz**.

2.1 Provability Logic

The most widely used provability logic is called **GL** after Gödel and Löb. As axioms, it contains all axiom schemes from **K** and the extra scheme GL:

$$\text{All (instances of) propositional tautologies} \tag{A1}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{A2}$$

$$\Box(\Box\varphi \to \varphi) \to \Box\varphi \tag{GL}$$

The rules of inference of **GL** are modus ponens and necessitation (if **GL** $\vdash \varphi$, then **GL** $\vdash \Box\varphi$). Note that **GL** $\vdash \Box\varphi \to \Box\Box\varphi$, as first proved by De Jongh and Sambin [1,13], but that the reflexivity axiom $\Box\varphi \to \varphi$ does not follow. Indeed, Segerberg proved in 1971 that provability logic is sound and complete with respect to all finite, transitive, irreflexive frames [11].

[2] In the rest of this paper, we take Φ to be finite, although the results can be extended to enumerably infinite Φ by the methods described in [8].

2.2 Grzegorczyk logic

Grzegorczyk Logic **Grz**, first introduced in [7], has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by Grz:

$$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{Grz}$$

Grz is sound and complete with respect to the class of all finite transitive, reflexive and anti-symmetric frames [1, Chapter 12].

2.3 Weak Grzegorczyk logic

Weak Grzegorczyk Logic **Grz** has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by wGrz, in which $\Box^+\psi := \Box\psi \wedge \psi$:

$$\Box^+(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{wGrz}$$

wGrz is sound and complete with respect to the class of all finite transitive, anti-symmetric frames (which need be neither irreflexive nor reflexive) [10].

3 Zero-one laws over relevant classes of finite models

3.1 GL: 0-1 law in finite irreflexive transitive models

We provide an axiomatization for almost sure model validity with respect to the relevant finite models corresponding to **GL**, namely the irreflexive transitive ones. The axiom system $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{GL}}$ has the same axioms and rules as **GL** plus:

$$\Box\Box\Box\bot \tag{T3}$$

$$\Diamond\top \to \Diamond A \tag{C1}$$

$$\Diamond\Diamond\top \to \Diamond(B \wedge \Diamond C) \tag{C2}$$

In the axiom schemes C1 and C2, the formulas A, B and C all stand for consistent conjunctions of literals over Φ. [3] Note that $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{GL}}$ is not a normal modal logic, because one cannot substitute just any formula for A, B, C. [4]

Definition 3.1 Define $M^{\Phi}_{GL} = (W, R, V)$, the *canonical asymptotic Kripke model* over Φ, with W, R, V as follows (see Fig. 1):
$W = \{b_v, m_v, u_v \mid v \text{ a propositional valuation on } \Phi\}$;
$R = \{\langle b_v, m_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle m_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle d_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\}$; and
for all $p_i \in \Phi$, the valuation V is defined by $V_{b_v}(p_i) = 1$ iff $v(p_i) = 1$; $V_{m_v}(p_i) = 1$ iff $v(p_i) = 1$; $V_{u_v}(p_i) = 1$ iff $v(p_i) = 1$.

The zero-one law for model validity now follows:
Theorem 3.2 *For every formula $\varphi \in L(\Phi)$, the following are equivalent:*

(i) $M^{\Phi}_{GL} \models \varphi$;

[3] C1 and C2 have been inspired by Carnap's consistency axiom: $\Diamond\varphi$ for any φ that is a consistent propositional formula [2], and used by Halpern and Kapron [8] for axiomatizing almost sure model validities for **K**-models.

[4] For example, substituting $\bot$ for A in C1 would make C1 equivalent to $\Box\bot$.

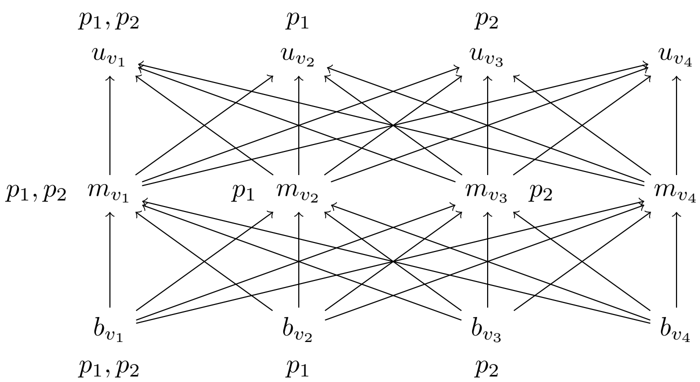

Fig. 1. The canonical asymptotic Kripke model $M^{\Phi}_{GL} = (W, R, V)$ for $\Phi = \{p_1, p_2\}$, defined in Definition 3.1. The accessibility relation is the transitive closure of the one drawn in the picture. The figure shows only propositional atoms true at each world.

(ii) $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{GL}} \vdash \varphi$;

(iii) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$;

(iv) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) \neq 0$.

3.2 Grz: 0-1 law in finite reflexive transitive anti-symmetric models

Define axiom system $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{Grz}}$ as **Grz** plus the following axioms:

$$\neg(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond(\neg\psi \wedge \chi \wedge \Diamond\neg\chi))) \tag{D3}$$

$$(\varphi \wedge \Diamond\neg\varphi) \to \Diamond A \tag{C3}$$

$$(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond\neg\psi) \to \Diamond(B \wedge \Diamond C) \tag{C4}$$

In the axiom schemes above, φ, ψ, χ stand for any formulas in $L(\Phi)$, while A, B and C stand for consistent conjunctions of literals over Φ. [5]

Definition 3.3 Define the canonical asymptotic Kripke model $M^{\Phi}_{Grz} = (W, R, V)$, where:
$W = \{b_v, m_v, u_v \mid v \text{ a propositional valuation on } \Phi\}$;
$R = \{\langle w, w\rangle \mid w \in W\} \cup$
$\{\langle b_v, m_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle m_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle d_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\}$; and
$V_{b_v}(p) = 1$ iff $v(p) = 1$; $V_{m_v}(p) = 1$ iff $v(p) = 1$; $V_{u_v}(p) = 1$ iff $v(p) = 1$.

Note that M^{Φ}_{Grz} is just the reflexive closure of M^{Φ}_{GL} (Definition 3.1).

Theorem 3.4 *For every $\varphi \in L(\Phi)$, the following are equivalent: (i) $M^{\Phi}_{Grz} \models \varphi$; (ii) $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{Grz}} \vdash \varphi$; (iii) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$; (iv) $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) \neq 0$.*

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[5] The axioms D3, C3 and C4 have been inspired by the axioms proposed in [8, Theorem 4.16] for the almost sure validities in finite **S4** models.

3.3 wGRz: 0-1 law in finite transitive anti-symmetric models

Define the axiom system $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{wGrz}}$ as **wGrz** plus axioms D3, C3 and C4.

Definition 3.5 The canonical asymptotic Kripke model M^{Φ}_{wGrz} is a combination of the irreflexive transitive M^{Φ}_{GL} and the reflexive transitive antisymmetric M^{Φ}_{Grz} (Def. 3.1 and 3.3), having a reflexive and irreflexive copy of each valuation-related world in each layer; it is transitive and antisymmetric and has direct accessibility from all states in the bottom layer to all states in the middle layer and all states in the middle layer to all states in the top layer.

Theorem 3.6 *For every $\varphi \in L(\Phi)$, the following are equivalent: (i) $M^{\Phi}_{wGrz} \models \varphi$; (ii) $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{wGrz}} \vdash \varphi$; (iii) $\lim_{n\to\infty} \nu_n(\varphi) = 1$; (iv) $\lim_{n\to\infty} \nu_n(\varphi) \neq 0$.*

Conclusion

We have formulated zero-one laws for provability logic, Grzegorczyk logic and weak Grzegorczyk logic, with respect to model validity. On the way, we have axiomatized validity in almost all relevant finite models, leading to three axiom systems. [6] Many questions are left open for future research, most notably, those about almost sure frame validity.

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[6] Proofs can be found in the full paper.